Allen A. Shaw

SOLID GEOMETRY

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SOLID GEOMETRY

By

W. H. MACAULAY, M.A.

Fellow of King's College
Cambridge

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PREFACE

THE limitations of the coordinate geometry in this book are stated at the beginning of the second chapter. A reader who has studied with care any such book as Salmon's *Conic Sections* will understand fairly well, without more explanation, the general character of further developments in geometry of three dimensions. Anyone who has not done this may ignore them.

The last four chapters are dictated to me by practical experience, extending over many years. I do not know now where to find satisfactory discussions of the geometry of strain, and of vector distributions, to which I could conveniently refer a pupil, for the purpose of supplementing and consolidating what he has learnt from more elementary and fragmentary treatment of these subjects.

With regard to terminology, it is convenient to be able to use a single word to express the relation between rectangular triads of the same type; and I have chosen, without authority, the word "conformable", (§ 8), the natural meaning of which expresses what is wanted. It could be defined so as to be applicable to triads which are not rectangular. I have employed several slightly unusual terms in chapter xv, see p. 251, but they have no claim to be called new. I have explained, (§ 185), Sir Robert Ball's convenient word "nole", though I have not ventured to use it.

Most of the examples are taken from Cambridge examination papers. Questions marked "C" are taken from College examination papers; and questions marked "S" are

taken from tripos papers, those taken from the papers of Part I of the Mathematical Tripos since 1910 being distinguished by the mark "S 1". Some questions derived from these sources are not marked because they have been slightly altered.

I am greatly indebted to Mr Arthur Berry for his advice and help, and I have to thank Mr H. W. Richmond for help with regard to some special points.

W. H. M.

October 12, 1930.

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CHAPTER I

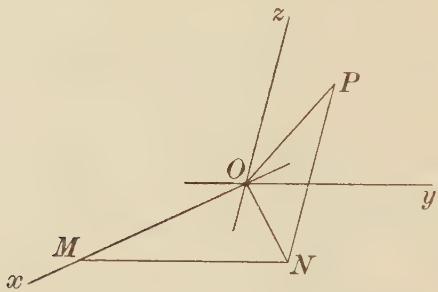
COORDINATES

1. In plane coordinate geometry the ordinary Cartesian coordinates, denoted by x and y, and measured with reference to a pair of axes, either rectangular or oblique, have the foremost place in the subject. A corresponding system of coordinates, also called Cartesian, holds a similar position in solid geometry, dealing with Euclidean space of three dimensions.

It may be constructed as follows. Choose a point, O, as origin, and two straight lines through it as axes of x and y, a positive direction being chosen along each of these lines. We then have x and y coordinates, as defined in plane geometry, for all points in the plane of these lines. This plane is called the plane of xy. Let us now choose a third straight line through O, not in this plane, and choose a positive direction along it, and call this line the axis of z; so that we have now a set of three axes of x, y and z, with positive directions Ox, Oy, Oz. The plane containing the axes of y and z is called the plane of yz, and the plane containing the axes of z and x is called the plane of zx.

In a diagram in which these axes are drawn take any point, P, whose position with reference to them is to be defined. Through P draw a straight line parallel to the axis of z, meeting the plane of xy in N. Then the position of P with reference to the axes is determined by the coordinates, x, y, of the point N, and the algebraical distance of P from N, reckoned positive when NP is the positive direction of the axis of z. Denoting this by z, we have three numbers, x, y and z, (the choice of a unit of length being assumed), which specify the position of P with reference to the axes. They are called its Cartesian coordinates. Each of them is a real number, positive or

negative or zero. They are represented in the diagram by OM, MN and NP. This figure can be drawn whatever the signs of the coordinates may be, M being the point whose coordinates are $(x, 0, 0)$, and N the point whose coordinates are $(x, y, 0)$. The three coordinates of a point are all on the same footing, for each of them is the algebraical distance of the point, measured parallel to one of the axes, from a plane drawn through the other two.



The axes may be drawn in two different ways. The way adopted here may be described by saying that the direction Oz makes a right-handed screw with the direction of rotation, about the axis of z, from Ox towards Oy, see § 185. The alternative way would be obtained by interchanging Ox and Oy. For any purpose of geometry it is immaterial which scheme is used. In physics the adoption of a standard scheme is sometimes a convenience.

2. The three coordinate planes, namely the planes of yz, zx and xy, divide space into eight regions, which are called octants. The signs of the coordinates of a point determine the octant in which it lies. If we take any point, with coordinates (a, b, c), there are seven other points whose coordinates have the same numerical values, if none of these are zero. Their coordinates are

$$(-a, b, c),\ (a, -b, c),\ (a, b, -c),\ (a, -b, -c),$$
$$(-a, b, -c),\ (-a, -b, c),\ (-a, -b, -c).$$

If any of these coordinates are zero, some of the points coincide. It is also to be noted that any set of three real numbers, adopted respectively as the values of x, y and z, defines, with reference to the axes, one and only one point.

If the axes are at right angles to one another they are called rectangular axes, and coordinates referred to them may be called rectangular coordinates. The coordinate planes are then at right angles to one another, and the coordinates of a point are its perpendicular algebraical distances from these planes. It is usual to assume, and it will be assumed here, that any coordinate axes which are used are rectangular when they are not otherwise specified, or unless it is obvious that no restriction is needed. Axes which are not rectangular are called oblique axes.

3. To find the distance from the origin of a point P, (x, y, z), (that is to say a point P whose coordinates are x, y, z), let M be the point $(x, 0, 0)$ on the axis of x, and N the point $(x, y, 0)$ in the plane of xy, and let r be the positive length OP. The axes are assumed to be rectangular, therefore PN is perpendicular to the plane of xy, and NM is perpendicular to the axis of x. Therefore

$$ON^2 = OM^2 + MN^2 = x^2 + y^2,$$

and $$OP^2 = ON^2 + z^2,$$

therefore $$r^2 = x^2 + y^2 + z^2.$$

The advantage of using rectangular axes will be seen by referring to the corresponding formula, (§ 14), for oblique axes.

To find the distance between any two points, P and Q, in terms of their coordinates, (x_1, y_1, z_1) and (x_2, y_2, z_2), draw through P a set of axes, PX, PY, PZ, in the same directions as the given axes, Ox, Oy, Oz; and let (X_2, Y_2, Z_2) be the coordinates of Q with reference to these axes. Then

$$PQ^2 = X_2^2 + Y_2^2 + Z_2^2.$$

But $$x_2 = x_1 + X_2, \quad y_2 = y_1 + Y_2, \quad z_2 = z_1 + Z_2,$$

therefore $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

4. *Orthogonal Projection.* The foot of the perpendicular from a given point on a plane is called its projection, or orthogonal projection, on the plane. Similarly the projection of a point on a straight line is the foot of the perpendicular from the point on the line. Thus, if the axes shown in the diagram, (§ 1), are rectangular, N is the projection of P on the plane of xy, and M is the projection of P on the axis of x.

The projection of a given figure on a plane means the figure formed by the points which are the projections of the points of the given figure. This use of the term projection, applied to geometrical figures, must be distinguished from the use of it in the sense of the resolved part of a step, (§ 10).

The area of the orthogonal projection of any plane area, A, on another plane, is $A \cos \alpha$, where α is the acute angle between the two planes. This is proved by dividing the given area into narrow strips by lines at right angles to the line of intersection of the two planes. The widths of the strips are not altered by the projection, and their lengths are all altered in the ratio of 1 to $\cos \alpha$.

5. *Directions and Direction Cosines.* All directions can be specified by radii drawn from some one point. Two radii, AP and BQ, of equal length, drawn from different points, A and B, and not in one straight line, have the same direction if $APQB$ is a parallelogram. Along any straight line there are two opposite directions. The term "sense" is sometimes used to express the distinction between them. Thus a certain direction along a straight line may be compared with a direction along a parallel line by saying that they have the same sense, or that they have opposite senses.

The angles between the directions of intersecting straight lines are defined in trigonometry. The angles between straight lines which do not intersect are defined as meaning

the corresponding angles between intersecting lines drawn parallel to them.

Let l, m, n be the cosines of the angles which a given radius makes with the positive directions of the axes, assumed to be rectangular. However these angles may be measured, according to the definition of an angle in trigonometry, l, m, n are a unique set of three numbers. They are called the direction cosines of the direction of the radius.

Let r be the positive length of a radius, OP, drawn from the origin to a point P, (x, y, z), and (l, m, n) its direction cosines; and let K be the projection of P on the axis of z. Then n is the cosine of the angle POz, which is equal to $\frac{OK}{OP}$ or $-\frac{OK}{OP}$ according as the angle is acute or obtuse. If the angle is acute $z = OK$, if it is obtuse $z = -OK$. Therefore in both cases $z = nr$. Similarly $x = lr$, and $y = mr$. Thus l, m, n are the coordinates of the point on the radius which is at a unit distance from the origin, and therefore specify its direction without ambiguity. And

$$r^2 = x^2 + y^2 + z^2,$$

therefore

$$l^2 + m^2 + n^2 = 1.$$

Also any set of three real numbers, l, m, n, which satisfy the equation $l^2 + m^2 + n^2 = 1$, will serve as a set of direction cosines; because there must be a point, P, at unit distance from the origin, whose coordinates are l, m, n, and therefore a direction OP with these cosines.

Also any set of three real numbers, a, b, c, if they are not all zero, could specify a direction, namely that of the radius drawn from the origin to the point (a, b, c). But when a, b, c are not direction cosines, a direction (a, b, c) will be understood here to mean either of the two opposite directions whose direction cosines are respectively

$$(ka, kb, kc) \text{ and } (-ka, -kb, -kc),$$

where $1/k$ is the square root of $a^2 + b^2 + c^2$. And if it is

unnecessary to distinguish between these two opposite directions, the direction (a, b, c) will be called a signless direction.

Thus a straight line regarded as a geometrical figure has a signless direction. A plane may be said to have a signless direction, because its orientation is specified by either of the two directions at right angles to it. A stretch, (§ 199), has a signless direction. There is no direction $(0, 0, 0)$. But there may, of course, be a direction which is the limiting case of a direction (a, b, c) when a, b and c tend to zero.

6. To find the angle, ϕ, between two given directions whose direction cosines are (l, m, n), (l', m', n'), draw radii, OP, OQ, of unit length, from the origin in these directions. Then (l, m, n), (l', m', n') are the coordinates of P and Q respectively. Therefore

$$PQ^2 = (l' - l)^2 + (m' - m)^2 + (n' - n)^2$$
$$= 2 - 2(ll' + mm' + nn').$$

Now if ϕ is the positive angle, less than π, between OP and OQ,

$$PQ = 2 \sin \tfrac{1}{2}\phi,$$

therefore $\qquad PQ^2 = (2 \sin \tfrac{1}{2}\phi)^2 = 2(1 - \cos \phi),$

which gives $\qquad \cos \phi = ll' + mm' + nn'.$

But all the angles between two given directions have the same cosine; therefore this formula is correct however the angle ϕ may be measured, by any rotation of a radius about O, in the plane POQ, from the position OP to the position OQ, or from the position OQ to the position OP.

Thus the condition that the given directions, OP, OQ, are at right angles is

$$ll' + mm' + nn' = 0.$$

This may be written

$$aa' + bb' + cc' = 0$$

when the directions are specified by (a, b, c) and (a', b', c'), which are not the actual direction cosines, but only proportional to them.

The formula for $\cos\phi$ gives

$$\begin{aligned}\sin^2\phi &= 1 - (ll' + mm' + nn')^2 \\ &= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2 \\ &= (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.\end{aligned}$$

If ϕ is restricted to being a positive angle less than π, $\sin\phi$ is the positive value of the square root of this expression. The sign of $\sin\phi$ is not specified by the two given directions, unless the way in which the angle ϕ is measured is in some way restricted.

7. The direction cosines, (λ, μ, ν), of a line at right angles to both the given directions (l, m, n) and (l', m', n'), are given by the equations

$$\begin{aligned} l\lambda + m\mu + n\nu &= 0, \\ l'\lambda + m'\mu + n'\nu &= 0, \\ \lambda^2 + \mu^2 + \nu^2 &= 1. \end{aligned}$$

The solution of these equations is

$$\frac{\lambda}{mn' - m'n} = \frac{\mu}{nl' - n'l} = \frac{\nu}{lm' - l'm} = \pm\frac{1}{\sin\phi},$$

where ϕ is the angle between the two given directions. The last fraction is obtained by taking the square root of the sum of the squares of the numerators and denominators of the other three. This gives two alternative directions, opposite to one another, at right angles to the given directions. No result is obtained when $mn' - m'n$, $nl' - n'l$ and $lm' - l'm$ are all zero, that is to say when the two given directions either coincide or are opposite to one another.

If the direction cosines of the two given directions are (l, m, n) and $(l + \delta l,\ m + \delta m,\ n + \delta n)$, inclined to one an-

other at an angle $\delta\theta$, and OP, OQ are unit radii in these directions,

$$PQ^2 = \delta l^2 + \delta m^2 + \delta n^2;$$

and we get the formula

$$(2 \sin \tfrac{1}{2} \delta\theta)^2 = \delta l^2 + \delta m^2 + \delta n^2,$$

which is useful when $\delta\theta$ is a small angle. In the limiting case in which δl, δm, δn and $\delta\theta$ are infinitesimals it gives

$$d\theta^2 = dl^2 + dm^2 + dn^2.$$

The direction cosines of PQ are proportional to δl, δm, δn; and the direction (λ, μ, ν), at right angles to OP and OQ, may be determined by the fact that it is at right angles to OP and PQ; therefore

$$\frac{\lambda}{m\delta n - n\delta m} = \frac{\mu}{n\delta l - l\delta n} = \frac{\nu}{l\delta m - m\delta l}.$$

This also is chiefly useful when $\delta\theta$ is a small angle.

8. *Conformability with the axes.* Let the two given directions, OP, (l, m, n), and OQ, (l', m', n'), be at right angles to one another, and let OR, (λ, μ, ν), be a direction at right angles to them both. Then $\sin^2 \phi = 1$, therefore the values of the direction cosines, λ, μ, ν, are either

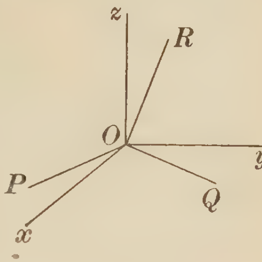
$$mn' - m'n, \quad nl' - n'l, \quad lm' - l'm,$$

or $$-(mn' - m'n), \quad -(nl' - n'l), \quad -(lm' - l'm).$$

These two cases may be distinguished from one another by comparing the directions OP, OQ, OR with the directions of the axes. Three directions OP, OQ, OR, at right angles to one another, taken in this order, will be said to be conformable with the axes if the arrangement of them is such that they can be shifted bodily into coincidence with the positive directions of the axes, OP coinciding with Ox, OQ with Oy and OR with Oz. The alternative arrangement is that which would make the direction OR opposite to the direction Oz, when OP coincides with Ox and OQ with Oy. Using this terminology, the case in which the direction

cosines of OR are $mn' - m'n$, $nl' - n'l$, $lm' - l'm$ is that in which OP, OQ, OR, taken in this order, are conformable with the axes Ox, Oy, Oz.

This can be established by showing that, if OP, OQ, OR are conformable with the axes, $lm' - l'm$ is positive when ν is positive, and negative when ν is negative. To prove this take OP, OQ, OR conformable with the axes, and of unit length, and ν positive. And let (r, θ), (r', θ') be the polar coordinates, in the plane of xy, of the projections of P and Q on this plane, the angles being measured from the axis of x towards the axis of y. The Cartesian coordinates of P and Q are (l, m, n), (l', m', n'); therefore



$$l = r\cos\theta, \quad m = r\sin\theta, \quad l' = r'\cos\theta', \quad m' = r'\sin\theta'.$$

Therefore $$lm' - l'm = rr'\sin(\theta' - \theta).$$

And P and Q are obviously arranged so that $\theta' - \theta$ is a positive angle less than π; therefore $lm' - l'm$ is positive. But when ν is negative P and Q must be interchanged in order to maintain conformability, and $lm' - l'm$ is negative.

9. *Reflection.* The term reflection will be used to describe the relation between two figures when each is the reflection of the other in a plane mirror.

Take any given figure, not all in one plane, consisting of points $P_1, P_2, \ldots\ldots$; and take any set of rectangular axes, with reference to which the coordinates of these points are

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad \ldots\ldots;$$

then the figure which consists of corresponding points $Q_1, Q_2, \ldots\ldots$, whose coordinates are respectively

$$(-x_1, y_1, z_1), \quad (-x_2, y_2, z_2), \quad \ldots\ldots,$$

is the reflection of the given figure with regard to the plane of yz.

It will now be proved that we get in this way a second figure of definite form, the same for all positions of the given figure with regard to the plane. This follows from the obvious fact that the distance between any two points of the given figure is equal to the distance between the corresponding points of the new figure. This shows that if the data for the construction of a figure are the distances between all pairs of points of it, there is an ambiguity as to the form of the figure; for the figure that is required and its reflection both satisfy the given conditions. But there is no other ambiguity as to the form of the figure; for if we construct the figure, point by point, an ambiguity which affects the form of the figure first arises when we have to fix the position of a fourth point which is not in the plane of the first three. There are two positions for it which satisfy the data, one on each side of the plane of the first three points. When one of these positions has been chosen, the construction can be completed without any further uncertainty, because there cannot be more than one point at given distances from four points which are not in one plane.

It will be noticed that if two sets of directions at right angles to one another, each taken in a certain order, are not conformable, each may be said to be the reflection of the other.

In solid geometry, a figure consisting of points all in one plane is the same as its reflection. The only reason why it appears to be different, when viewed in a mirror, is that we then get a view of it as seen from the other side of the plane. A given configuration of points cannot be shifted bodily so as to be brought into coincidence with a figure which is its reflection, so that corresponding points coincide, except when it is in one plane.

10. *Projection of Steps.* A portion of straight line, specified by its length and direction, may be called a step. The length is treated as an algebraical number, with the

interpretation that a step of length a, with direction cosines (l, m, n), is the same as a step of length $-a$, with direction cosines $(-l, -m, -n)$. Thus either of the two opposite directions along the line of a given step may be taken as the direction of the step, the distinction between them being provided by the sign of the length. This agrees, of course, with the way in which a force or a velocity is specified in mechanics.

A step of length a, with direction cosines (l, m, n), may be called the step (a, l, m, n). The term step is introduced here for the sake of clearness, but it is commonly replaced by the word line when there is no ambiguity as to the nature of the specification.

Let (λ, μ, ν) be the direction cosines of a given direction. Then

$$a(l\lambda + m\mu + n\nu)$$

is called the projection of the step (a, l, m, n) on a line in the given direction. It is the product of the length of the step and the cosine of the angle between the given direction and that of the step. Thus it is the algebraical distance, in the direction (λ, μ, ν), through which a point would move in travelling along the step in question. In the more convenient language of mechanics, it would be called the resolved part of the step in the direction (λ, μ, ν). It will be noticed that it is not affected by a reversal of the signs of all the four numbers a, l, m, n.

If P and Q are two given points, a step PQ means a step with positive length PQ and direction from P to Q. Starting from any point, P, draw a succession of steps,

$$PQ, QR, RS, \ldots\ldots KP,$$

forming a closed polygon $PQR \ldots\ldots K$, in general not in one plane. If a point traverses the complete circuit of this polygon, the total algebraical distance through which it moves, measured in any given direction, (λ, μ, ν), is zero. That is to say the sum of the projections of the steps

on a line in any given direction is zero. This result may be written

$$\Sigma a\,(l\lambda + m\mu + n\nu) = 0;$$

where Σ denotes summation for all the steps, (a_1, l_1, m_1, n_1), (a_2, l_2, m_2, n_2) ......, which form the closed polygon.

We may also use the equivalent statement that the projection of the step PK is equal to the sum of the projections of the steps PQ, QR ...... up to K.

This is well known as the geometrical proposition from which the law of composition and resolution of forces in equilibrium is derived. And the simplest way of using it in geometry is to resolve the steps, forming a polygon, in any convenient direction or directions, as if they were forces in equilibrium. The word resolution suggests the procedure more clearly than the word projection. Any mention of projection of one line on another must be understood to imply that account is taken of a direction along each line, as distinguished from the opposite direction.

Take as an example the polygon $OMNPO$, shown in the diagram, (§ 1), OM, MN, NP being parallel to the axes. Let the axes be oblique; the angles yOz, zOx, xOy, between their positive directions, being denoted respectively by ϕ, χ, ψ. And let (x, y, z) be the coordinates of P, r the positive length OP, and α, β, γ the angles which OP makes with positive directions of the axes. Here we have not got direction cosines, but all the angles involved are given. Projection of the sides of the polygon, namely OM, MN, NP, PO, on the line OP, gives

$$x \cos \alpha + y \cos \beta + z \cos \gamma - r = 0;$$

and projection on the axis of x gives

$$x + y \cos \psi + z \cos \chi - r \cos \alpha = 0.$$

These results are independent of the signs of the coordinates of P. OM, for example, is regarded as a step whose direction is the positive direction of the axis of x, and whose length, x, may be positive or negative. On

the other hand, r is defined as positive, and the angles α, β, γ specify the direction of the step OP.

11. If (x_1, y_1, z_1), (x_2, y_2, z_2) are the rectangular coordinates of two points, P and Q, the projections of PQ on the axes are

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1.$$

The points P and Q can be joined by a zigzag of lines, PM parallel to the axis of x, MN parallel to the axis of y, and NQ parallel to the axis of z, so that $PMNQP$ is a closed polygon. And the projection of PQ on any given line is equal to the sum of the projections of PM, MN and NQ on the same line. Now PM has length $x_2 - x_1$, (positive or negative), and direction (1, 0, 0), and MN has length $y_2 - y_1$ and direction (0, 1, 0), and NQ has length $z_2 - z_1$ and direction (0, 0, 1). Therefore the projection of PQ on a line whose direction cosines are (λ, μ, ν) is equal to

$$\lambda (x_2 - x_1) + \mu (y_2 - y_1) + \nu (z_2 - z_1).$$

If PQ has length a and direction cosines (l, m, n), this is also

$$a (l\lambda + m\mu + n\nu),$$

and $$x_2 - x_1 = la, \quad y_2 - y_1 = ma, \quad z_2 - z_1 = na.$$

12. *Equations of a Straight Line.* A straight line, in the sense of a geometrical figure, a line of unlimited length, may be specified by the coordinates, (x', y', z'), of a point A on the line, and the direction cosines, (l, m, n), of one of the two opposite directions along it.

This point and this direction being given, any other point, P, on the line is specified by its algebraical distance from A, which will be denoted by ρ, reckoned as positive when AP is the direction (l, m, n), and negative when it is the opposite direction $(-l, -m, -n)$. Thus if (x, y, z) are the coordinates of P we have

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho.$$

These equations, which give the coordinates of any point

on a given straight line in terms of ρ, are called the parametric equations of the line, and ρ is called a parameter.

To regard the position of a point on the line as specified by the value of ρ at that point, and to treat the coordinates of the point as functions of ρ, is often more convenient than taking one of the coordinates as the independent variable.

By eliminating ρ we can get two equations, involving x, y and z, which are an alternative set of equations of the line, being satisfied at all points on the line, and at no other points. If l, m and n are none of them zero, these equations are

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}.$$

This leads to the consideration of a straight line as the intersection of two planes, see § 35.

Some examples of the use of a set of parametric equations to represent a curve occur in plane geometry. The equations $x = a\cos\phi$, $y = b\sin\phi$, representing an ellipse, referred to its principal axes, are a case of this. Here ϕ, the eccentric angle, is a parameter, which may be adopted as the independent variable instead of one of the coordinates. There are also curves, such as a cycloid, for which the Cartesian coordinates of a point would always be expressed in terms of a parameter. In solid geometry, with three Cartesian coordinates, this procedure is more obviously useful for the sake of symmetry. The introduction of a third coordinate makes it necessary to study symmetry, and to pay attention to treating the coordinates as being on an equal footing. The use of three direction cosines, instead of two angles, for the specification of a direction, is an example of this attention to symmetry. The introduction of a parameter in the case of a straight line may appear to be unnecessary, but the symmetry which is thus secured is sometimes important.

13. The parametric equations of a given straight line, namely

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

may also be written

$$x = x' + a\sigma, \quad y = y' + b\sigma, \quad z = z' + c\sigma,$$

where $l = ka$, $m = kb$, $n = kc$ and $\sigma = k\rho$, and k is any number. This is a more general form of a set of parametric equations of a straight line, representing a line drawn through a point (x', y', z'), with direction (a, b, c); σ being a parameter each value of which corresponds to a point on the line.

Let a straight line be specified by the coordinates, (x_1, y_1, z_1), (x_2, y_2, z_2), of two given points on it, A and B. Its direction cosines are proportional to

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1;$$

therefore

$$x = x_1 + (x_2 - x_1)\,\sigma, \;\; y = y_1 + (y_2 - y_1)\,\sigma, \;\; z = z_1 + (z_2 - z_1)\,\sigma$$

are a set of parametric equations of the line. Every value of σ gives the coordinates of a point on the line. The value 0 gives A, and the value 1 gives B, and for any other point, P, (x, y, z), on the line,

$$\sigma = \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Therefore σ is equal to the ratio of $\overline{AP}$ to $\overline{AB}$, where $\overline{AP}$ and $\overline{AB}$ denote the algebraical distances of the points P and B from A, measured in the direction AB.

These equations can be put into a more symmetrical form by writing λ/μ for the ratio in which the segment AB, of the straight line, is divided, either internally or externally, by the point P. That is to say λ/μ is the ratio

of $\overline{AP}$ to $\overline{PB}$, where $\overline{AP}$ and $\overline{PB}$ denote algebraical distances measured in the direction AB. Therefore

$$\frac{\lambda}{\mu} = \frac{\overline{AP}}{\overline{AB} - \overline{AP}} = \frac{\sigma}{1-\sigma},$$

therefore $$\frac{\lambda}{\lambda+\mu} = \sigma, \quad \text{and} \quad \frac{\mu}{\lambda+\mu} = 1 - \sigma,$$

and the equations take the form

$$x = \frac{\mu x_1 + \lambda x_2}{\lambda + \mu}, \quad y = \frac{\mu y_1 + \lambda y_2}{\lambda + \mu}, \quad z = \frac{\mu z_1 + \lambda z_2}{\lambda + \mu}.$$

These equations give the coordinates of a point which divides AB in the ratio of λ to μ, λ and μ being any given numbers, positive or negative or zero. They are also a set of parametric equations of the straight line AB, the parameter being λ/μ.

14. *Oblique Coordinates.* Some attention must be paid to the formulae for oblique axes. If (x, y, z) are the coordinates of a point P at a distance r from the origin, the ratios x/r, y/r, z/r, though no longer cosines, define the direction OP. They may be called direction ratios with reference to oblique axes, and may be denoted by l, m, n.

With oblique axes we still have a point $Q\,(x_2, y_2, z_2)$, reached from a point $P\,(x_1, y_1, z_1)$ by three steps,

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1,$$

in the directions of the axes. And if a straight line is drawn through a given point (x', y', z'), with given direction ratios l, m, n, the parametric equations of the line are

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

where ρ is the algebraical distance of a point, (x, y, z), on the line, from the point (x', y', z'). And we still have the equations

$$x = x_1 + (x_2 - x_1)\,\sigma, \quad y = y_1 + (y_2 - y_1)\,\sigma, \quad z = z_1 + (z_2 - z_1)\,\sigma,$$

for the coordinates of any point on the straight line through (x_1, y_1, z_1) and (x_2, y_2, z_2). Also the equations

$$x = \frac{\mu x_1 + \lambda x_2}{\lambda + \mu}, \quad y = \frac{\mu y_1 + \lambda y_2}{\lambda + \mu}, \quad z = \frac{\mu z_1 + \lambda z_2}{\lambda + \mu}.$$

The distance, r, of a point P from the origin, in terms of its oblique coordinates, x, y, z, can be calculated as follows.

Let ϕ, χ, ψ be the angles yOz, zOx, xOy, between the positive directions of the axes; and let α, β, γ be the angles which the direction OP makes with the axes. The projection of OP on the axis of x is $r \cos \alpha$. But this is also equal to the sum of the projections of the steps by which P can be reached from O, namely x, y and z in the directions of the three axes respectively. Therefore, as in § 10,

$$r \cos \alpha = x + y \cos \psi + z \cos \chi.$$

Similarly
$$r \cos \beta = x \cos \psi + y + z \cos \phi,$$

and
$$r \cos \gamma = x \cos \chi + y \cos \phi + z.$$

Also r is equal to the sum of the projections on OP of the same three steps, that is to say

$$r = x \cos \alpha + y \cos \beta + z \cos \gamma.$$

Substituting in this equation the values found for $\cos \alpha$, $\cos \beta$ and $\cos \gamma$, we get

$$r^2 = x^2 + y^2 + z^2 + 2yz \cos \phi + 2zx \cos \chi + 2xy \cos \psi.$$

Substituting lr, mr and nr for x, y and z, we get the relation which every set of direction ratios with reference to oblique axes must satisfy, namely:

$$l^2 + m^2 + n^2 + 2mn \cos \phi + 2nl \cos \chi + 2lm \cos \psi = 1.$$

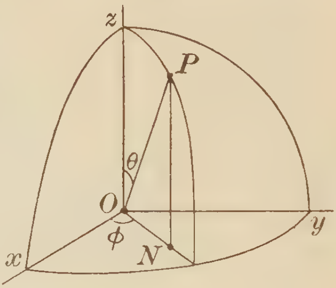
15. *Polar Coordinates.* Several different sets of polar coordinates of a point may be used. The simplest are r, l, m, n, representing the positive length and the direction

cosines of the radius, OP, drawn from the origin to the point in question. Thus

$$x = lr, \quad y = mr, \quad z = nr,$$

x, y, z being the rectangular coordinates of P.

Another standard set of polar coordinates is obtained by expressing the direction cosines of OP in terms of two angles, θ and ϕ. Draw a sphere through P, with centre O. Let θ be the angle, between 0 and π, which OP makes with Oz; and let ϕ be the angle, between 0 and 2π, which ON makes with Ox, measured in the direction from Ox to Oy, N being the foot of the perpendicular from P on the plane of xy. Then, r being the positive length OP, $z = r\cos\theta$, and $ON = r\sin\theta$. Therefore



$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Thus r, θ, ϕ are a set of polar coordinates of P. The limits proposed for the angles provide for each point having a single set of coordinates. This may be unimportant.

The angles θ and ϕ may be regarded as specifying a point on a given sphere, and may then be called the polar distance and the longitude of the point.

Another set of coordinates, called cylindrical coordinates, are r, θ, z, where r, θ are the polar coordinates, in plane geometry, of the projection of the point on the plane of xy. Thus $x = r\cos\theta$, $y = r\sin\theta$.

These are simple examples of sets of coordinates which may for special reasons be employed. But the main development of the theory of coordinate geometry has depended on the use of coordinates which are either Cartesian, or have linear relations with Cartesian coordinates.

16. *Examples.*

1. A cube rests on a plane inclined at an angle α to a horizontal plane, one diagonal of the base lying along a line of greatest slope. Find the inclinations to the vertical of the edges and diagonals of the cube. (S 1.)

2. Find the angle between the line $x = \sqrt{3}y,\ z = 0$ and the line

$$\sqrt{6}x = 2\sqrt{2}y = 2\sqrt{3}z.$$

3. The coordinates of the middle points of the sides of a triangle ABC being given, find the lengths and direction cosines of BC, CA and AB.

4. Find the direction of a straight line drawn from the origin to cut the line

$$(x-a)/l = (y-b)/m = (z-c)/n$$

at right angles.

5. Three diagonals of a rectangular parallelepiped make with the fourth diagonal acute angles θ, ϕ, ψ. Prove that, provided the sum of the squares of the two shorter edges exceeds the square of the longest edge,

$$\cos\theta + \cos\phi + \cos\psi = 1. \qquad \text{(S 1.)}$$

6. O, P, P' are three points on the line $y = 0,\ z = c$; and O', Q, Q' are three points on the line $x = 0,\ z = -c$; O, O' being on the axis of z. Prove that, if PQ is perpendicular to $P'Q'$,

$$OP.OP' + O'Q.O'Q' + OO'^2 = 0,$$

$OP, OP', O'Q, O'Q'$ being taken positive in the directions of the axes. (S 1.)

7. Perpendiculars are drawn from one corner of a rectangular block upon the three diagonals which do not pass through that corner. Prove that, if these perpendiculars are projected on any edge of the block, one of the projections is equal to the sum of the other two. (S 1.)

8. AOA', BOB', COC' are three diameters of the earth, regarded as a sphere of radius a; C is the north pole, and A, B are points on the equator having longitudes 0° and 90°. OA, OB, OC are taken as axes of x, y, z, and P is a point on the surface having latitude λ and longitude ϕ. The line $A'P$ is produced to cut the plane $x = a$ in Q. Find the coordinates of P and Q; and prove that if P describes a parallel of latitude λ, Q describes the circle

$$x = a, \quad y^2 + z^2 - 4az \operatorname{cosec}\lambda + 4a^2 = 0. \qquad \text{(S 1.)}$$

9. The equations of four straight lines are

$$(1)\ y = a,\quad z = a'; \qquad (2)\ z = b,\quad x = b';$$

$$(3)\ x = c,\quad y = c'; \qquad (4)\ x = y = z.$$

Find the equations of the line drawn from the point (k, k, k) to cut the lines (1) and (2). Prove that two straight lines can be drawn to cut all the four given lines; and that these lines cut the line (4) in two points determined by the equation

$$(k - a)(k - b)(k - c) = (k - a')(k - b')(k - c'). \qquad \text{(S 1.)}$$

10. Find the angles between the three straight lines drawn from one corner of a cube to the middle points of the opposite faces; also the angle between one of these lines and a line at right angles to each of the others.

11. A straight line is drawn through point (a, b, c), and intersecting each of the straight lines

$$\frac{x - \alpha}{\lambda} = \frac{y - \beta}{\mu} = \frac{z - \gamma}{\nu} \quad \text{and} \quad \frac{x - \alpha'}{\lambda'} = \frac{y - \beta'}{\mu'} = \frac{z - \gamma'}{\nu'}.$$

Calculate its direction cosines.

12. Prove that the three straight lines joining the middle points of opposite edges of a tetrahedron bisect one another; and that the point in which these lines meet is on the line joining any angular point of the tetrahedron to the centre of gravity of the opposite face.

13. Referred to oblique axes, the coordinates of P and Q are $(a, b, 0)$ and $(0, 0, c)$. Find the projection of PQ on the axis of x; the angles, ϕ, χ, ψ, between the positive directions of the axes being given.

CHAPTER II

INTERPRETATION OF EQUATIONS

17. *Locus of an Equation.* Coordinate Geometry is based on the interpretation of equations. The signification of an equation, $f(x, y, z) = 0$, is that it specifies all the sets of values of x, y and z which satisfy it. The sets of values which are real numbers, positive or negative or zero, are interpreted as the coordinates of real points, each of which has its place in a figure. And the sets of values which involve i, or $\sqrt{(-1)}$, are interpreted as the coordinates of imaginary points, which have no place in a figure. Thus the complete signification of the equation is that it specifies a certain aggregate of points, either real or imaginary. This aggregate of points is called the locus of the equation; and the aggregate of real points which it specifies is called its real locus.

We shall consider here only real loci, ignoring imaginary points; and the term locus will be used in the sense of real locus when this does not cause ambiguity.

It must be noted that, by the use of Cartesian coordinates, points at infinity are distinguished from other points; because we have a point, namely the origin, which cannot be at infinity. Thus parallel planes are distinguished from intersecting planes, and cylinders from cones. And there is no inconvenience in referring to parallel planes as planes which do not meet, as points at infinity do not possess the significance which they have in projective geometry.

18. The character of the limitations adopted here, and the way in which geometrical theory is developed and consolidated by the use of imaginary as well as real points, together with a more symmetrical system of coordinates, can be understood from experience of plane geometry.

To see examples of this, in plane geometry, it is not

necessary to look further than the theory of conics. The distinction between an ellipse and a hyperbola is a thing which concerns real points, and is obscured by the consideration of imaginary points. But the property of circles, in one plane, that they have two common points at infinity, is a thing which concerns imaginary points. And this leads to general properties of conics with two points in common, which may be real, and need not be at infinity. It is only by treating real and imaginary points as being on an equal footing that full advantage can be taken of algebraical theory. But occasional and inconsistent reference to imaginary points, in connection with a study of real figures, is not very helpful.

19. Algebraical equations are classed according to their degree, or order. Degree and order are used indifferently, as equivalent terms. The general equations of the first and second orders will be written

$$Ax + By + Cz + D = 0,$$

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$+ 2ux + 2vy + 2wz + d = 0.$$

The equations of the third and higher orders can similarly be written at length. The equation of the first order is also called a linear equation.

It will be assumed that algebraical equations are all written in the form
$$\Sigma A x^{\alpha} y^{\beta} z^{\gamma} = 0,$$

where α, β and γ are positive whole numbers or zero, and the coefficients, represented here by A, are real numbers. The order of an equation is given by the greatest value of $\alpha + \beta + \gamma$ which occurs in it. If $\alpha + \beta + \gamma$ has the same value for all terms of an equation, the equation is said to be homogeneous.

In the general theory of coordinate geometry, in which the coordinates may be any numbers, either real or imaginary, it

is usual to assume that all the coefficients in equations whose locus is studied are real numbers.

Any equation, whether algebraical or not, is said to be homogeneous if its signification depends only on the ratios, $x : y : z$, of the coordinates.

20. To construct the locus, that is to say the real locus, of a given equation, $f(x, y, z) = 0$, we may regard it as an equation which gives one of the coordinates which occur in the equation, say z, in terms of the other two; and may erect at each point of the plane of xy, for which z has one or more real values, an ordinate or ordinates to represent these values of z. The extremities of these ordinates are the real points whose coordinates satisfy the equation. The locus thus constructed is called a surface. It agrees in general with our primitive conception of a surface if $f(x, y, z)$ is a continuous function of x, y and z, possessing differential coefficients. Algebraical equations satisfy this condition.

It is obvious that there are equations which have no real locus; for example the equation

$$x^2 + y^2 + z^2 + 1 = 0$$

has no real locus.

There are cases in which the real locus of an equation is very scanty, so that it would not be called a surface except for the sake of consistent terminology. For example the real locus of the equation

$$2x^2 + 3y^2 = 0$$

is a single straight line, the axis of z. Here so far as the real locus is concerned the equation is equivalent to a pair of equations, $x = 0$, $y = 0$. Similarly the real locus of the equation

$$ax^2 + by^2 + cz^2 = 0,$$

where a, b and c are any positive numbers, is a single point, namely the origin.

The locus of a pair of equations is the aggregate of points whose coordinates satisfy both the equations. This is, in

general, a curve, namely the curve in which the surfaces represented by the two equations intersect. Any such locus can presumably be represented by other pairs of simultaneous equations in a variety of ways. It may be useful to arrange the two equations so that each of them involves only two of the coordinates. The curve is then represented by its "plan" and "elevation".

A curve may also be represented by a set of three parametric equations, (§ 12). Elimination of the parameter would then give two equations involving only the coordinates.

Three simultaneous equations are, in general, sufficient for finding the values, or several sets of values, of x, y and z, and thus, in general, represent one point or several distinct points, the points of intersection of three surfaces. For example, the point (a, b, c), given by the equations $x = a$, $y = b$, $z = c$, is the point of intersection of the three planes which these equations represent.

21. *Sections of a Surface.* The curve in which a given plane cuts a surface is called the section of the surface by that plane.

The sections of a surface, $f(x, y, z) = 0$, by a series of planes parallel to one of the coordinate planes, provide a method of tracing the form of the surface which has the advantage of giving a graphical representation of it. The section by a plane $z = c$ is the plane curve whose equation in x and y coordinates is $f(x, y, c) = 0$. And if the sections are drawn for all values of c they comprise all points on the surface.

The contour lines drawn on a map, for certain given intervals of height, are an example of the practical use of a series of sections for showing the shape of a surface. The shape could be represented, in this way, to any required degree of accuracy, by taking the intervals of height small enough, the contour lines being correspondingly numerous, and supplemented by some indication of the distinction between hills and hollows.

22. *Tangent plane, and Developable surfaces.* The tangent at any point of a curve which is a plane section of a given surface is called a tangent line of the surface at that point. A straight line which lies on a surface is also reckoned as a tangent line of the surface at each point of its length. In general the tangent lines at a given point of a curved surface are in one plane. This is called the tangent plane of the surface at the point in question; and the straight line, through the same point, at right angles to the tangent plane is called the normal. A point on the surface at which there is no such plane is called a singular point. The vertex of a cone is a singular point, being a point at which there is no tangent plane, but the tangent planes at all other points of the cone pass through it.

By reference to § 167 it will be seen that, if $F(x, y, z) = 0$ is the equation of a surface, the direction cosines of the normal at any point of it are proportional to the values, at that point, of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$; and that a singular point on the surface is a point at which these three numbers are all zero.

In plane geometry, if a straight line moves so as to be a tangent of a given curve, the curve is called the envelope of this line. The line is conceived to move continuously through a single series of positions, and is therefore said to have one degree of freedom, its equation involving one variable parameter. For example, the parabola $y^2 = 4ax$ is the envelope of the straight line represented by the equation

$$y = mx + \frac{a}{m},$$

in which m is a parameter each value of which specifies a position of the line.

Similarly, if a plane moves so as to be the tangent plane of a given surface, the surface is called the envelope of the plane. In general the motion of the plane, in contact with a given surface, is that which is described by saying that the plane has two degrees of freedom; in other words, its equation involves two parameters which may be varied

independently. The plane is not required to move through a single series of consecutive positions. For example, a tangent plane of a sphere may start to move into a new position by rocking about any tangent line through its point of contact.

There is however a class of surfaces whose tangent planes, like the tangents of a curve, have only one degree of freedom. They are called developable surfaces. Thus a developable surface may be defined as the envelope of a plane represented by an equation involving only one variable parameter. The tangent plane has only a single series of consecutive positions, and therefore touches the surface along a straight line, about which it must rock in order to begin to move into a new position.

It follows from this that a developable surface has straight lines on it about which it might be bent, if made of flexible material, without any other distortion, so as to be developed into a plane.

23. *Ruled Surface.* A surface which can be traced out, (or generated), by a moving straight line, (as a curve can be traced out by a moving point), is called a ruled surface; and the straight lines which thus lie on a ruled surface are called its generating lines, or rectilinear generators. A surface may have more than one system of such lines. And there may be straight lines on a surface which do not belong to a system of generators.

A developable surface is a ruled surface for which each generating line is the line of contact of a tangent plane.

A ruled surface which is not a developable is called a skew surface, or a scroll.

The following example will show how a ruled surface is generated by a straight line, moving with one degree of freedom.

Let the line move so that it always intersects the parabola

$$x^2 = az, \quad y = 0,$$

and also intersects the straight line

$$x + y = 0, \quad z = 0,$$

at right angles. These equations show that, for any value of λ, $(\lambda, 0, \lambda^2/a)$ is a point on the parabola; and that, for any value of μ, $(\mu, -\mu, 0)$ is a point on the given straight line. And

$$\frac{x-\mu}{\lambda-\mu} = \frac{y+\mu}{\mu} = \frac{az}{\lambda^2}$$

are the equations of the straight line through these two points. To provide for this line being at right angles to the given straight line, whose direction is $(1, -1, 0)$, we have

$$\lambda - \mu - \mu = 0, \quad \text{or} \quad \lambda = 2\mu.$$

Thus the equations of the moving straight line are

$$x - \mu = y + \mu = az/4\mu.$$

Each value of μ gives a single position for this line, which therefore moves with one degree of freedom. And the result of eliminating μ between these two equations, namely

$$x^2 - y^2 = az,$$

is the equation which is satisfied at all points on the line, in all its positions, and is therefore the equation of the surface generated.

24. *Cone and Cylinder.* A cone and a cylinder are examples of developable surfaces. A cone is a surface which can be generated by lines drawn through a point, called the vertex of the cone. And a cylinder is a surface which can be generated by parallel lines.

Accordingly a cylinder may be regarded as the limiting case of a cone when its vertex passes to infinity. But in geometry based on Cartesian coordinates the two surfaces can be distinguished. When a more general system of coordinates is used, analogous to trilinear coordinates in plane geometry, the distinction may cease to be convenient, and in that case may be dropped as naturally as it is here adopted.

It is obvious that a plane, or a pair of intersecting planes, is both a cone and a cylinder. If a pair of intersecting planes is counted as a cone, any point on the line of intersection is a vertex. A single straight line may be counted as a cone, any point on it being taken as vertex; or it may be counted as a cylinder, with generating lines coincident.

It is useful to note that a cone for which each of two points, O and O', is a vertex is also a cylinder. It may be the single straight line OO'. And if a point P, not on the line OO', is on the surface, each of the straight lines OP and $O'P$ is on the surface, and consequently the whole plane OPO'. Similarly, if another point, P', not on the plane OPO', is on the surface, the whole plane $OP'O'$ is on the surface. Therefore the surface is either the single straight line, OO', or else a set of planes intersecting in the line OO'.

Simple forms of the equations of a cylinder and a cone can easily be obtained. In the case of a cylinder, take one of the coordinate axes, say the axis of z, parallel to the generating lines. Then the surface is fully specified by its section by the plane of xy. Let $f(x, y) = 0$ be the equation, in x and y coordinates, of this section. Then $f(x, y) = 0$, an equation which does not involve z, is the equation of the surface generated by straight lines drawn parallel to the axis of z through every point of the section. Thus every equation in which one of the coordinates is absent is the equation of a cylinder, if it has a real locus.

If the equation, $f(x, y) = 0$, is of the second order, the section by the plane of xy is a conic; and the surface is called a circular, or elliptic, or hyperbolic, or parabolic cylinder, according as the section is a circle, or an ellipse, or a hyperbola, or a parabola. If the conic is a pair of straight lines the surface is a pair of planes. If it is a single point the surface is a single straight line.

In the case of a cone take a vertex for origin. Any homogeneous algebraical equation, (§ 19), is satisfied at the origin, O. And if it is satisfied at any other point,

P, (x', y', z'), it is satisfied at every point whose coordinates are (kx', ky', kz'), that is to say every point on the straight line OP. Thus the real locus of the equation is either a single point, namely the origin, or else a cone with a vertex at the origin. And conversely, the equation of a cone with vertex at the origin must be one which is satisfied at the origin, and whose signification elsewhere depends only on the ratios $x : y : z$, that is to say it is homogeneous. And this is true whether the equation is algebraical or not.

A cone with a single vertex can be specified by its vertex and one section by a plane not passing through the vertex. The following example will show how the equation of a cone so specified can be found.

Let (α, β, γ) be the coordinates of the vertex, and let the given section be the circle $x^2 + y^2 = a^2$, $z = 0$, in the plane of xy. Let (l, m, n) be the direction cosines of a generating line. Then the equations of this line, (§ 12), are

$$x = \alpha + l\rho, \quad y = \beta + m\rho, \quad z = \gamma + n\rho.$$

The condition that it intersects the circle is expressed by the equations

$$(\alpha + l\rho)^2 + (\beta + m\rho)^2 = a^2, \qquad \gamma + n\rho = 0;$$

and elimination of ρ between these two equations gives

$$(\alpha n - \gamma l)^2 + (\beta n - \gamma m)^2 = a^2 n^2.$$

But if (x, y, z) are the coordinates of any point on the cone other than the vertex, the direction cosines of the generating line through this point are proportional to $x - \alpha$, $y - \beta$, $z - \gamma$. Therefore

$$\{\alpha (z - \gamma) - \gamma (x - \alpha)\}^2 + \{\beta (z - \gamma) - \gamma (y - \beta)\}^2 = a^2 (z - \gamma)^2.$$

And this equation is also satisfied at the vertex, therefore it is the equation of the cone. It may be written

$$(\alpha z - \gamma x)^2 + (\beta z - \gamma y)^2 = a^2 (z - \gamma)^2.$$

25. It can be proved, (see § 106), that the equation of any cone which can be represented by an equation of the second order, and has a single vertex, can be put into the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

And it is obvious that this equation represents a cone with vertex at the origin, whose sections by planes parallel to the plane of xy are similar ellipses, the section by the plane $z=k$ being the ellipse

$$x^2/a^2+y^2/b^2=k^2/c^2, \quad z=k.$$

Assuming the proposition, (§ 28), that an equation of the first order represents a plane, it will be seen that the equation of the tangent plane of this cone at a point (x', y', z') is

$$x'x/a^2+y'y/b^2-z'z/c^2=0\,;$$

for this is an equation of the first order, which is satisfied at the origin, and at all points on the straight line

$$x'x/a^2+y'y/b^2=z'^2/c^2, \quad z=z',$$

which we know from conics to be the tangent line at the point (x', y', z') of the elliptic section by the plane $z=z'$.

The equation of the cone in polar coordinates, r, l, m, n, is

$$r^2\,(l^2/a^2+m^2/b^2-n^2/c^2)=0,$$

which is equivalent to $l^2/a^2+m^2/b^2-n^2/c^2=0$ at all points except the origin. Thus a homogeneous relation between direction cosines specifies the surface if the vertex is known.

26. *Change of sign of a Function.* Let $f(x, y, z)$ be a continuous function of the coordinates, which has a single value, real and finite, at every point, though it may tend towards an infinite value at infinity. If it has a positive value at a point A, and a negative value at a point B, every path drawn from A to B must contain at least one point at

which it is zero, because it must change continuously along this path from being positive to being negative. Thus A and B must be situated so that every path from A to B cuts the surface $f(x, y, z) = 0$. This shows that, if this surface divides space into several distinct regions, the values of the function at all points in any one of these regions have the same sign. And if the function changes its sign there must be this division into regions. This supplies a test of whether two given points are on the same or opposite sides of some given surface, or portion of surface. But attention must be paid to the conditions to be satisfied by the function $f(x, y, z)$. They are always satisfied when $f(x, y, z) = 0$ is an algebraical equation written in the standard form

$$\Sigma A x^\alpha y^\beta z^\gamma = 0, \quad (\S\ 19).$$

Take, as an example, the equations

$$xy - 1 = 0 \quad \text{and} \quad y - \frac{1}{x} = 0,$$

which represent the same surface, a hyperbolic cylinder. The rule is applicable to the function $xy - 1$; but it is not applicable to the function $y - \dfrac{1}{x}$, which is discontinuous when $x = 0$. The cylinder divides space into three regions; and $xy - 1$ is negative in one of them, and positive in the other two.

27. *Examples.*

1. Find the locus of a point whose distance from the point (1, 2, 3) is half its distance from the point (2, -3, 1).

2. What surfaces are represented by the equations

$$x^2 + y^2 = 1, \quad x^2 + y^2 = 4z, \quad x^2 + y^2 = 4z^2\,?$$

3. Find the equation of a cylinder each of whose generating lines is equally inclined to the three axes, and whose section by the plane of xy is the conic $ax^2 + by^2 = 1$, $z = 0$. [$a(x - z)^2 + b(y - z)^2 = 1$.]

4. Find the equation of a right circular cone with vertex at the point (1, 1, 0), axis parallel to the axis of z, and vertical angle 2α.

5. Show that, through any given point, (α, β, γ), on the surface $zx = y$, two and only two straight lines can be drawn so as to lie on the surface; and find their direction cosines. [Seek the directions (l, m, n) for which every point of the line $x = \alpha + l\rho$, $y = \beta + m\rho$, $z = \gamma + n\rho$ is on the surface.]

6. Find the equation of the surface, (anchor ring), generated by the revolution of a circle about a straight line in its plane.

7. Of a set of three straight lines at right angles to one another, drawn through a fixed point, one is in the plane of zx, and another is in a plane through the axis of z equally inclined to the axes of x and y. Find the equation of the surface generated by the third.

CHAPTER III

PLANES AND STRAIGHT LINES

28. *The Equation of a Plane.* The surface represented by an equation of the first order, (or linear equation),

$$Ax + By + Cz + D = 0,$$

referred either to rectangular or oblique axes, is a plane. To prove this, take any two points, (x', y', z') and (x'', y'', z''), on the surface; then the coordinates of any point on the straight line drawn through them may be written

$$x' + \sigma (x'' - x'), \quad y' + \sigma (y'' - y'), \quad z' + \sigma (z'' - z'),$$

(§ 13); and we have the equations

$$Ax' + By' + Cz' + D = 0,$$
$$Ax'' + By'' + Cz'' + D = 0.$$

Multiplying the first of these equations by $1 - \sigma$, and the second by σ, and adding, we get

$$A \{x' + \sigma (x'' - x')\} + B \{y' + \sigma (y'' - y')\} + C \{z' + \sigma (z'' - z')\} + D = 0.$$

Therefore every point on every straight line drawn through two points on the surface is on the surface. This shows that the surface is a plane, if it is not a single straight line. And examination of the sections of the surface by the coordinate planes shows that it cannot be a single straight line.

The coefficients can be chosen so that the equation represents any given plane; for a plane may be specified by three points on it, not in one straight line, and if the coordinates of these points are given, we have three equations from which the ratios to one another of A, B, C and D can be found. If the plane passes through the origin D is zero.

Taking any plane, $Ax + By + Cz + D = 0$, which does not pass through the origin, the equation of the parallel plane through the origin is $Ax + By + Cz = 0$, for the planes represented by these equations have no point in common. And as planes which are parallel to the same plane are parallel to one another, the two planes

$$Ax + By + Cz + D = 0,$$

$$A'x + B'y + C'z + D' = 0$$

are parallel or coincident if the ratios to one another of A, B and C are the same as the ratios to one another of A', B' and C'.

29. A plane which does not pass through the origin can be specified by the points in which it cuts the axes. If it cuts all the axes, the points of section being $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

because this is a linear equation which satisfies the conditions. Here a, b and c are called the intercepts on the axes. If the plane cuts two of the axes, say the axes of x and y, in points $(a, 0, 0)$, $(0, b, 0)$, and is parallel to the third, the equation is

$$\frac{x}{a} + \frac{y}{b} = 1$$

for the same reason. If it cuts only one axis, say the axis of x, being parallel to the plane of yz, the equation is $x = a$.

30. *Rectangular Axes.* Let us now restrict the axes to being rectangular, so that we have direction cosines and other conveniences. A plane may be specified by the positive length, p, and the direction cosines, (l, m, n), of the perpendicular, OY, drawn to it from the origin. And a point P, (x, y, z), is on the plane if, and not unless, the projection of the line OP on the line OY is equal to p.

Now this projection is $lx + my + nz$, because P is reached from O by steps x, y and z in the directions of the axes, (§ 10). Therefore the equation of the plane is

$$lx + my + nz = p.$$

This equation is correct when the origin is on the plane, and p consequently zero; but the direction, (l, m, n), of the line on which the projection is made, may in this case be either of the two opposite directions at right angles to the plane.

Any linear equation,

$$Ax + By + Cz + D = 0,$$

can be put into the form

$$lx + my + nz = p,$$

without altering its signification, by dividing each coefficient by k, where

$$k^2 = A^2 + B^2 + C^2,$$

k being chosen so that $-D/k$ is positive if it is not zero. If D is zero k may be either positive or negative. An equation so treated is said to be rectified. The coefficients of x, y and z become a set of direction cosines, and the length of the perpendicular on the plane from the origin is $-D/k$.

The orientation of the plane with reference to the axes is given by the signless direction (A, B, C), (§ 5).

The same procedure of projection could be applied if the axes were oblique, giving the equation of a plane in the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

where p is the length of the perpendicular from the origin, and α, β, γ are its angles of inclination to the axes. But this would not be a useful form of the equation, because the divisor, k, which would put any given equation of a plane into this form, would involve the angles between the axes in a complicated way.

As an example of rectification, consider the plane

$$x + 2y - 2z + 6 = 0.$$

The equation is rectified by dividing by -3, and thus becomes

$$-\tfrac{1}{3}x - \tfrac{2}{3}y + \tfrac{2}{3}z = 2.$$

This shows that the length of the perpendicular drawn from the origin is 2, and that its direction cosines are $(-\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3})$. The orientation of the plane is given by the signless direction $(1, 2, -2)$.

31. *Perpendicular distance from a Plane.* The length of the perpendicular from any point, P, (x', y', z'), to a plane whose rectified equation is

$$lx + my + nz = p,$$

can be found as follows. Draw a straight line through P in the direction whose direction cosines are (l, m, n). The equations of this line are

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

(§ 13); and the value of ρ for the point in which it meets the plane is given by the equation

$$l(x' + l\rho) + m(y' + m\rho) + n(z' + n\rho) = p,$$

or

$$\rho = p - lx' - my' - nz'.$$

The value of ρ thus obtained is the algebraical distance from P to the plane, measured in the direction (l, m, n), at right angles to the plane. It is positive when P is on one side of the plane, and negative when P is on the other side. If the origin is not on the plane, the positive side is that on which the origin lies. If the origin is on the plane the two sides can be distinguished by means of the coordinates of some other known point. In all cases the length of the perpendicular from P is the positive value of

$$\pm(p - lx' - my' - nz').$$

This shows that, if the equation of the plane is given in the form

$$Ax + By + Cz + D = 0,$$

the algebraical distance of a point (x', y', z') from it is

$$\frac{1}{k}(Ax' + By' + Cz' + D),$$

where $$k^2 = A^2 + B^2 + C^2.$$

Here k may be taken to be either positive or negative; and the side of the plane on which this expression is positive can then be distinguished from the side on which it is negative by reference to the origin, or some other known point. Two points, $(x'\ y', z')$ and (x'', y'', z''), are on the same side of the plane if

$$Ax' + By' + Cz' + D \text{ and } Ax'' + By'' + Cz'' + D$$

are both positive or both negative, and are on opposite sides if one is positive and the other negative, (§ 26).

32. The equations of any straight line drawn through any point P, (x', y', z'), with direction cosines (λ, μ, ν), are

$$x = x' + \lambda\rho, \quad y = y' + \mu\rho, \quad z = z' + \nu\rho.$$

And the value of ρ for the point in which this line meets a plane,

$$Ax + By + Cz + D = 0,$$

is given by the equation

$$A(x' + \lambda\rho) + B(y' + \mu\rho) + C(z' + \nu\rho) + D = 0,$$

or $$(A\lambda + B\mu + C\nu)\rho + Ax' + By' + Cz' + D = 0.$$

The value of ρ given by this equation is the algebraical distance from the point P to the plane, measured in the direction whose direction cosines are (λ, μ, ν). The formula for the perpendicular distance from P to the plane is a particular case of this.

33. *Planes drawn through a given straight line.* Taking

$$Ax + By + Cz + D = 0$$

and $$A'x + B'y + C'z + D' = 0$$

as the equations of two intersecting planes, the equation of any other plane through their line of intersection may be written

$$Ax + By + Cz + D + k(A'x + B'y + C'z + D') = 0;$$

for this is the equation of a plane, and is satisfied at every point common to the two given planes. By suitable choice of a value for k it can be made to satisfy one other condition. The plane can be made to pass through any given point, (x', y', z'), by the adoption of the value for k which is given by the equation

$$Ax' + By' + Cz' + D + k(A'x' + B'y' + C'z' + D') = 0.$$

Or it can be made parallel to one of the coordinate planes by choosing k so as to make either $A + kA'$ or $B + kB'$ or $C + kC'$ zero. This is the same thing as the result of eliminating one of the coordinates between the two given equations in the usual way.

Assuming that the axes are rectangular, the equations of the two planes which bisect the angles between the two given planes are

$$\frac{Ax + By + Cz + D}{\sqrt{(A^2 + B^2 + C^2)}} = \pm \frac{A'x + B'y + C'z + D'}{\sqrt{(A'^2 + B'^2 + C'^2)}};$$

for these equations express an obvious property of each point on the bisecting planes, namely that the lengths of the perpendiculars from this point on the two given planes are equal. The plus sign gives one bisecting plane, and the minus sign gives the other. That they satisfy the test for being at right angles, (§ 6), can easily be verified.

A concise notation is sometimes convenient. Let $S, S', S'', \ldots\ldots$ be given linear functions of x, y and z, namely

$$Ax + By + Cz + D, \quad A'x + B'y + C'z + D', \ldots\ldots.$$

Then $S + \lambda S' = 0$ is the equation of a plane drawn through

the line of intersection of the planes $S = 0$ and $S' = 0$; and one more condition to be satisfied by the plane will determine λ. Similarly

$$S + \lambda S' + \mu S'' = 0$$

is the equation of a plane drawn through the point of intersection of the planes $S = 0$, $S' = 0$, $S'' = 0$; and two more conditions to be satisfied by the plane are needed for finding λ and μ.

34. *Intersections of three Planes.* In general three given planes intersect in a single point, whose coordinates are found by solving the equations of the planes. The exceptional cases, in which the planes do not meet or meet in more than one point, are those in which a plane can be drawn at right angles to each of the three given planes; for they may be classified as follows: (i) the case in which the three planes have a common line of intersection, the conditions for which have been given; (ii) the case in which two or more of the planes are parallel, the conditions for which we know; (iii) the case in which the three planes are arranged so as to be the sides of a triangular prism. In case (ii), if only two of the planes are parallel the third cuts them in two parallel lines, and a plane at right angles to these lines is at right angles to each of the planes.

Accordingly we have to find the condition that a plane can be drawn at right angles to each of three given planes. Let us take

$$\begin{aligned} Ax + By + Cz + D &= 0, \\ A'x + B'y + C'z + D' &= 0, \\ A''x + B''y + C''z + D'' &= 0, \end{aligned}$$

as the equations of these planes, and if possible draw a plane, $lx + my + nz = 0$, at right angles to each of them. This is possible if, and not unless, the three equations

$$\begin{aligned} Al + Bm + Cn &= 0, \\ A'l + B'm + C'n &= 0, \\ A''l + B''m + C''n &= 0, \end{aligned}$$

can be satisfied simultaneously by values of l, m, n which

are not all zero. Therefore the exceptional cases are those in which

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix} = 0.$$

If this condition is satisfied we know how to recognise cases (i) and (ii), and therefore also case (iii). If the numerical values of the coefficients of the equations are known, the three cases are easily distinguished by means of the traces of the planes on one of the coordinate planes.

35. *Equations of a Straight Line.* A straight line can be represented by any pair of simultaneous equations representing planes drawn through it. Thus the choice of a pair of equations to represent a given straight line can be made in any number of different ways. If $S = 0$, $S' = 0$ are a pair of linear equations representing a certain straight line, any pair of equations of the form

$$\lambda S + \mu S' = 0, \quad \lambda' S + \mu' S' = 0$$

represent the same straight line, and a convenient pair can be chosen.

A pair of equations can always be chosen, in this way, so that each equation contains not more than two of the coordinates, and therefore represents a plane parallel to one of the coordinate axes; a plane being said to be parallel to a given straight line if it does not meet it. Thus we can get a pair of equations of the type

$$y = \alpha x + \beta, \quad z = \gamma x + \delta.$$

Equations of this type exhibit the line as specified by four numbers, α, β, γ, δ. And we can prove that the number of independent quantities needed for the specification of a straight line is four; because two are needed for the specification of a signless direction, and two more are sufficient to specify a point in which the line meets some given plane, such as one of the coordinate planes.

Any pair of simultaneous equations of this type can be written, in various ways, in the parametric form

$$x = x' + a\rho, \quad y = y' + b\rho, \quad z = z' + c\rho,$$

pp 72, 13,

which exhibits the line as drawn through a point (x', y', z') with the signless direction (a, b, c); or so that a, b and c are actual direction cosines, (§ 5). Also if a, b and c are none of them zero, the equations may be written

$$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c};$$

or in terms of the coordinates, (x', y', z') and (x'', y'', z''), of two points on the line, in the form

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'} = \frac{z - z'}{z'' - z'}.$$

A universal form of the equations of a straight line through two given points, to which there is no exception, is the parametric form, (§ 13),

$$x = x' + (x'' - x')\,\sigma, \quad y = y' + (y'' - y')\,\sigma, \quad z = z' + (z'' - z')\,\sigma.$$

The angle between two straight lines, whose equations are given, can be found from their direction cosines, (§ 6).

As an example of the various forms in which the equations of a straight line may be written, take the line represented by the pair of equations

$$3x - 4y + z + 11 = 0, \quad 6x - 2y - z + 10 = 0.$$

Elimination of z gives $3x - 2y + 7 = 0$, and elimination of x gives $2y - z - 4 = 0$; therefore the line is represented by this pair of equations. They are the equations of planes drawn through the given line, and parallel respectively to the axis of z and the axis of x. They may be written in the form

$$\frac{x + 1}{2} = \frac{y - 2}{3} = \frac{z}{6}.$$

Therefore $(-1, 2, 0)$ is a point on the line, and the direc-

tion is $(2, 3, 6)$. If actual direction cosines, $(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$, are used, we get the set of parametric equations

$$x = -1 + \tfrac{2}{7}\rho, \quad y = 2 + \tfrac{3}{7}\rho, \quad z = \tfrac{6}{7}\rho.$$

By giving various values to ρ, the coordinates of any number of other points on the line can be found, and the form of the equations can be varied accordingly. For example the equations

$$\frac{x-1}{1+3} = \frac{y-5}{5+1} = \frac{z-6}{6+6}$$

exhibit the same line as drawn through the points $(1, 5, 6)$ and $(-3, -1, -6)$.

36. *The Six Coordinates of a Straight Line.* Three planes drawn through a given straight line, each of them parallel to one of the coordinate axes, give a symmetrical specification of the line.

Starting with

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

as the equations of the line, the equations of the three planes in question are

$$ny - mz = \lambda, \quad lz - nx = \mu, \quad mx - ly = \nu,$$

where $\quad \lambda = bn - cm, \quad \mu = cl - an, \quad \nu = am - bl.$

This shows that λ, μ, ν are connected with l, m, n by the identical relation

$$l\lambda + m\mu + n\nu = 0.$$

The six numbers l, m, n, λ, μ, ν are called the six coordinates of the straight line. It is obvious that they specify it. We are concerned only with the ratios to one another of these six numbers, and we have an identical relation connecting them, therefore this specification of the line involves four, and only four, independent quantities.

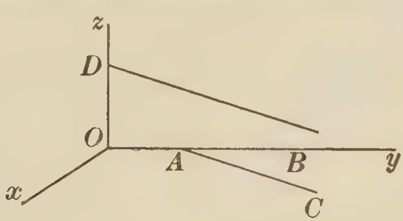
Also $a\lambda + b\mu + c\nu = 0$. Thus assuming, as is usual, that the axes are rectangular, we have three directions at right

angles to one another, namely (l, m, n), (λ, μ, ν) and that of the perpendicular from the origin on the line. (See example 26, in § 47.)

37. Two straight lines are parallel if they have the same two opposite directions. They are said to be coplanar if a plane can be drawn through them; this is the case if they intersect or are parallel.

The condition that two straight lines are coplanar can be found as follows. Let $S_1 = 0$, $S_2 = 0$ be the equations of one line, then $\lambda S_1 + \mu S_2 = 0$ is the general equation of a plane drawn through it. Similarly $\lambda' S_3 + \mu' S_4 = 0$ is the general equation of a plane drawn through another line $S_3 = 0$, $S_4 = 0$. And the condition that the lines are coplanar is that values can be found for λ/μ and λ'/μ' which make the equations $\lambda S_1 + \mu S_2 = 0$ and $\lambda' S_3 + \mu' S_4 = 0$ identical.

38. *Shortest Distance between Straight Lines.* If two given straight lines are not coplanar, let us take the first line, AB, for axis of y; and through any point, A, on this line, draw a line AC parallel to the second line, and take the plane CAB for plane of xy. This settles the plane of yz, a plane through AB at right angles to the plane CAB. The



second line must cut the plane of yz in some point, D. Take the plane of zx through this point. This settles the coordinate axes. The axis of z intersects both the given lines at right angles, and is the line of the shortest distance between them, for OD is the perpendicular distance between two parallel planes each containing one of the lines. The

equations of the first line are $x = 0$, $z = 0$; and the equations of the second line are $x = y \tan \alpha$, $z = k$, where α is the angle which the second line makes with the axis of y, and $(0, 0, k)$ are the coordinates of D.

A more symmetrical arrangement is to take as origin the middle point of the shortest distance between the lines, and to take the plane of yz so that the lines are equally inclined to it. The equations of the two lines then take the form

$$y = mx, \quad z = c; \quad y = -mx, \quad z = -c.$$

This is usually the simplest form in which the equations of any two given straight lines can be written, when we are at liberty to choose coordinate axes in the most symmetrical position with regard to them. They are applicable to any pair of straight lines; for they give intersecting lines if c is zero, and parallel lines if m is zero.

39. Having established the existence of a line of shortest distance between two straight lines which are not coplanar, let us now find its length and direction when no special choice is made of coordinate axes.

Let one line be drawn through a point A, (α, β, γ), with direction (a, b, c), and the other through a point B, $(\alpha', \beta', \gamma')$, with direction (a', b', c'); and let d be the length, and (λ, μ, ν) the direction, of the shortest distance. This direction is at right angles to each of the lines, therefore

$$a\lambda + b\mu + c\nu = 0,$$

$$a'\lambda + b'\mu + c'\nu = 0,$$

therefore $$\frac{\lambda}{bc' - b'c} = \frac{\mu}{ca' - c'a} = \frac{\nu}{ab' - a'b}.$$

And d is the projection on a line (λ, μ, ν) of either AB or BA; therefore d is the positive value of

$$\pm \frac{(\alpha - \alpha')(bc' - b'c) + (\beta - \beta')(ca' - c'a) + (\gamma - \gamma')(ab' - a'b)}{\sqrt{\{(bc' - b'c)^2 + (ca' - c'a)^2 + (ab' - a'b)^2\}}}.$$

The equations of the line of the shortest distance can also be found, for it is the line of intersection of a plane through A whose equation is

$$(b\nu - c\mu)(x - \alpha) + (c\lambda - a\nu)(y - \beta) + (a\mu - b\lambda)(z - \gamma) = 0,$$

because the perpendicular to it is at right angles to (a, b, c) and to (λ, μ, ν), and a plane through B whose equation is

$$(b'\nu - c'\mu)(x - \alpha') + (c'\lambda - a'\nu)(y - \beta') + (a'\mu - b'\lambda)(z - \gamma') = 0,$$

because the perpendicular to it is at right angles to (a', b', c') and to (λ, μ, ν).

The condition that two lines, specified as above, are coplanar is

$$(\alpha - \alpha')(bc' - b'c) + (\beta - \beta')(ca' - c'a) + (\gamma - \gamma')(ab' - a'b) = 0.$$

For this equation is satisfied if $bc' - b'c$, $ca' - c'a$ and $ab' - a'b$ are all zero, in which case the lines are parallel; and in every other case the meaning of the equation is that the shortest distance between the lines is zero.

40. The length of the perpendicular, PN, drawn from a point P, (f, g, h), on a straight line, AB, specified by its direction cosines, (l, m, n), and a point A, (α, β, γ), on the line, is the square root of $AP^2 - AN^2$, because PAN is a right angled triangle. Also AN is the projection of AP on the line AB. Therefore PN is the square root of

$$(f - \alpha)^2 + (g - \beta)^2 + (h - \gamma)^2 - \{l(f - \alpha) + m(g - \beta) + n(h - \gamma)\}^2.$$

41. The equation of a plane specified by three given points on it, namely (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), not in one straight line, may be found by the elimination of A, B, C and D between the four equations

$$Ax + By + Cz + D = 0,$$
$$Ax_1 + By_1 + Cz_1 + D = 0,$$
$$Ax_2 + By_2 + Cz_2 + D = 0,$$
$$Ax_3 + By_3 + Cz_3 + D = 0;$$

which gives

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0$$

Pl. thru 3 given pts

as the equation of the plane.

If the given points are in one straight line we get no result, the equation being satisfied for all values of x, y and z. The form of the determinant gives a proof of this, see § 13.

A simpler result is obtained by writing the equation of the plane in the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

and eliminating A, B and C between this and the two equations

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0,$$

$$A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) = 0.$$

The equation of the plane is thus obtained in the form

$$\begin{vmatrix} x - x_1, & y - y_1, & z - z_1 \\ x_2 - x_1, & y_2 - y_1, & z_2 - z_1 \\ x_3 - x_1, & y_3 - y_1, & z_3 - z_1 \end{vmatrix} = 0.$$

It is obvious, from the properties of determinants, that each of these equations satisfies the required conditions, as they are linear equations satisfied at each of the given points. They are correct for oblique as well as rectangular axes.

42. *Tetrahedron.* With rectangular axes, the determinant with four rows and columns leads to a proposition about the volume of a tetrahedron, in terms of the coordinates of its angular points, which corresponds to the proposition in plane geometry, with x and y coordinates,

the condition that 3 pts sh- lie in o

line is $\begin{vmatrix} x_1 - x_2, & y_1 - y_2 \\ x_2 - x_3, & y_2 - y_3 \end{vmatrix} = 0$

that the area of a triangle whose angular points are (x_1, y_1), (x_2, y_2), (x_3, y_3) is the positive value of p.l.n

$$\pm \frac{1}{2} \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}. = \pm \frac{1}{2} \begin{vmatrix} x_1 - x_2, & y_1 - y_2 \\ x_2 - x_3, & y_2 - y_3 \end{vmatrix} \text{ etc}$$

The equation of the plane drawn through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) may be written

$$\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix} x + \begin{vmatrix} z_1, & x_1, & 1 \\ z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \end{vmatrix} y + \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} z$$

$$= \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

Let us also write it

$$lx + my + nz = p.$$

Let A be the area of the triangle formed by the three given points; and let us write A_1, A_2, A_3 for lA, mA, nA respectively, so that the areas of the orthogonal projections of this triangle on the coordinate planes are the positive values of $\pm A_1$, $\pm A_2$, $\pm A_3$. And $A^2 = A_1^2 + A_2^2 + A_3^2$.

Accordingly the equation of the plane is

$$A_1 x + A_2 y + A_3 z = pA;$$

therefore there is a number, k, such that

$$A_1 = k \begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}, \quad A_2 = k \begin{vmatrix} z_1, & x_1, & 1 \\ z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \end{vmatrix},$$

$$A_3 = k \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \quad \text{and} \quad pA = k \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

And as the area of the projection of A on the plane of xy is either A_3 or $-A_3$, k is either $\frac{1}{2}$ or $-\frac{1}{2}$. Now the length of the perpendicular from a fourth point, (x_4, y_4, z_4), on the plane drawn through the three given points, is the positive value of

$$\pm \frac{1}{A}(A_1x_4 + A_2y_4 + A_3z_4 - pA).$$

Also we know that the volume of a tetradedron is one-third of the product of the area of its base and the length of the perpendicular from the vertex. Therefore the volume of the tetrahedron whose angular points are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) is the positive value of

$$\pm \tfrac{1}{3}(A_1x_4 + A_2y_4 + A_3z_4 - pA);$$

and this is equal to the positive value of

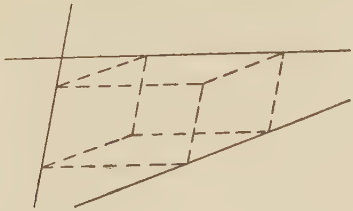
$$\pm \frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

This shows that the equation of the plane through three given points is an expression of the fact that, if any fourth point is taken in that plane, the volume of the tetrahedron with the four points for its angular points is zero.

43. *Oblique Axes.* There may very well be cases in which oblique axes are useful, and not troublesome. For example, if we have three given straight lines, no two of which are in one plane, and which are not all parallel to one plane, these lines can be taken as edges of a parallelepiped, and define this parallelepiped. Take the centre of the parallelepiped for origin, and coordinate axes parallel to the edges. The equations of the three given lines are then

$$y = b,\ z = -c;\quad z = c,\ x = -a;\quad x = a,\ y = -b.$$

The three given lines are shown in the diagram, and the other edges of the parallelepiped are indicated by dotted lines.



As an example of the use of axes thus chosen with regard to three given straight lines, consider the equation

$$ayz + bzx + cxy + abc = 0.$$

It represents a surface on which each line lies, for it is satisfied by $y = b$, $z = -c$, and by $z = c$, $x = -a$, and by $x = a$, $y = -b$. Now draw a straight line, PQ, intersecting each of the three given lines. This line must be in each of two planes whose equations may be written

$$z + c = \lambda(y - b), \text{ and } z - c = \mu(x + a);$$

and if it meets the third line, there is a point on the intersection of these two planes for which $x = a$ and $y = -b$, for which therefore

$$z = -c - 2\lambda b = c + 2\mu a.$$

Thus if PQ meets all the three lines, we have

$$b\lambda + a\mu + c = 0;$$

and the coordinates of any point on PQ, in any position of this line, satisfy the equation

$$b\frac{z + c}{y - b} + a\frac{z - c}{x + a} + c = 0,$$

or
$$ayz + bzx + cxy + abc = 0.$$

Thus the line PQ, moving so as to intersect the three given lines generates the surface represented by this equation. It will, of course, reach limiting positions in which it is

parallel to one of the given lines, the point of intersection passing to infinity.

To take another case in which it is natural to use oblique axes, consider the geometry of a tetrahedron. It is obvious that a plane section of a tetrahedron is either a triangle or a quadrilateral. Now if two planes intersect, a line in one which is parallel to a line in the other, must be parallel to their line of intersection. Thus a plane section of a tetrahedron is a parallelogram if, and not unless, the sides of the parallelogram are parallel respectively to two opposite edges of the tetrahedron. Let us find the condition that a section is a rhombus. Take three faces of the tetrahedron for coordinate planes, and let

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

be the equation of the plane forming the fourth face; and let d be the length of the edge which is opposite to the axis of z. Then the equation of a plane parallel to the edges c and d is

$$\frac{x}{\lambda a} + \frac{y}{\lambda b} = 1.$$

The section by this plane is a parallelogram $ABCD$, and $AB = \lambda d$, and $BC = (1 - \lambda) c$. Thus the section is a rhombus if $\lambda (c + d) = c$, that is to say if

$$AB = cd/(c + d).$$

44. In dealing with a question involving two given straight lines, it may be convenient to take two of the coordinate axes parallel to these lines, and the third axis along the line of the shortest distance between them. We get in this way a set of axes such that two of the angles between them are right angles, so that the formula, (§ 14), for the distance between two points is not unduly complicated. The equations of the two given straight lines may then be written $x = 0$, $z = c$, and $y = 0$, $z = -c$; the axes of x and y being taken parallel to them, and the origin half way between them.

45. *A Pair of Planes.* The surface represented by the equation

$$(Ax + By + Cz + D)(A'x + B'y + C'z + D') = 0$$

is a pair of planes, for the points at which it is satisfied are those which lie on either of the two planes

$$Ax + By + Cz + D = 0, \text{ and } A'x + B'y + C'z + D' = 0.$$

If the planes coincide the surface is called a pair of coincident planes, to mark the fact that it is represented by an equation of the second order.

In all cases the surface is a cylinder. If the planes intersect it is also a cone, such that any point on the line of intersection may be regarded as a vertex. And if the planes coincide it is a cone such that any point on the surface may be regarded as a vertex.

The equation may be simplified by suitable choice of coordinate axes. If the planes are parallel, the plane of xy may be taken parallel to them, and midway between them, and the equation is

$$(z - c)(z + c) = 0, \text{ or } z^2 - c^2 = 0.$$

If the planes intersect, the line of intersection may be taken for axis of z; and, with rectangular axes, the planes of yz and zx may be taken bisecting the angles between them, so that the equation is

$$(y - mx)(y + mx) = 0, \text{ or } y^2 - m^2x^2 = 0.$$

Each equation includes the case of a pair of coincident planes.

It is to be noted that the equation

$$ax^2 + by^2 + cz^2 = 0$$

cannot represent a pair of planes unless at least one of the coefficients is zero. For if they are all positive or all negative the equation is satisfied only at the origin; and if two are positive and one negative, or two negative and one positive, the equation represents a cone with elliptic sections, (§ 25).

46. If
$$Ax + By + Cz + D = 0$$
and
$$A'x + B'y + C'z + D' = 0$$
are the equations of planes which intersect, the real locus of the equation
$$(Ax + By + Cz + D)^2 + (A'x + B'y + C'z + D')^2 = 0$$
is a straight line, namely the line of intersection of the planes.

A straight line thus obtained may also be regarded as the limit of a cone of small angle, or of a circular, or elliptic, cylinder of small cross section, in the same way that a single point may be regarded as a sphere of zero radius.

47. *Examples.*

1. Find the shortest distance between an edge of a given cube and a diagonal which does not meet it.

2. Find the equations of the two planes through the points
$$(0,\ 4,\ -3),\quad (6,\ -4,\ 3),$$
other than the plane through the origin, which cut off from the axes intercepts whose sum is zero. (S 1.)

3. Show that the necessary and sufficient condition that the two straight lines
$$x = mz + a, \quad y = nz + b \quad \text{and} \quad x = m'z + a', \quad y = n'z + b'$$
intersect is $(a - a')(n - n') = (b - b')(m - m')$. Given that these lines intersect, find the equation of the plane in which they lie. (S 1.)

4. Investigate the nature of the intersections of each of the following four sets of three planes:

(i) $x + 2y + 3z - 6 = 0, \quad 3x + 5y + z - 5 = 0, \quad 2x + y + 2z - 3 = 0;$

(ii) $x - 2y + z - 4 = 0, \quad 5x - y - z - 10 = 0, \quad x + y - z - 7 = 0;$

(iii) $2x - 5y + z - 3 = 0, \quad x + y + z - 2 = 0, \quad 5x - 2y + 4z + 3 = 0;$

(iv) $x + y + z - 1 = 0, \quad 2x - 3y + 5z - 2 = 0, \quad 3x - y - z - 3 = 0.$

In (ii) find the area of the cross section of the prism of which the planes are faces. [The direction of the edges is found to be (1, 2, 3), and the area of the section by one of the coordinate planes can be calculated.]

5. The plane $3x + 2y + z = 33$ meets the straight lines

$$x = 2y = 8z, \quad x = y = 2z, \quad 4x = 7y = 7z$$

in A, B, C respectively. Find the distances OA, OB, OC, where O is the origin, and show that the normal to the plane makes with each of the lines OA, OB, OC an angle whose cosine is $11/\sqrt{126}$. (S 1.)

6. Find the perpendicular distance of the point (3, 1, 2) from the line of intersection of the planes

$$4x + 3y - z = 12, \quad 7x - 2y + 5z = 8. \qquad \text{(C.)}$$

7. Find the equation of the plane which contains the two parallel straight lines

$$(x-a)/l = (y-b)/m = (z-c)/n, \quad (x-a')/l = (y-b')/m = (z-c')/n.$$

8. The feet A, B, C of a tripod with legs of equal length are at the points (0, 0), (3, 9), (7, 1) referred to rectangular axes in a horizontal plane. The apex, P, of the tripod is at a height 12 above the plane. Find the cosines of the angles between (i) the lines AP and BC, (ii) the planes PAB and PAC. (S 1.)

9. A straight line whose direction cosines are (l, m, n) meets each of the lines $y = x\tan\alpha$, $z = c$ and $y = -x\tan\alpha$, $z = -c$. Find the length of the shortest distance between this line and the axis of z. Also show that the equations of the line of shortest distance are

$$lx + my = 0, \quad z(l^2 + m^2)\sin\alpha\cos\alpha + clm = 0. \qquad \text{(S 1.)}$$

10. The direction cosines of two intersecting straight lines being given, find the direction cosines of the two lines which bisect the angles between them.

11. Show that the length of the shortest distance between the line $z = x\tan\alpha$, $y = 0$ and any tangent of the ellipse $x^2\sin^2\alpha + y^2 = a^2$, $z = 0$ is constant. (S 1.

12. Prove that if $\alpha + \beta + \gamma = 0$, the line $x + \alpha = y + \beta = z + \gamma$ intersects at right angles each of the four lines

$$x = 0,\ y + z = 3\alpha; \quad y = 0,\ z + x = 3\beta; \quad z = 0,\ x + y = 3\gamma;$$

$$x + y + z = 3\lambda,\ \frac{\alpha x}{\lambda - \alpha} + \frac{\beta y}{\lambda - \beta} + \frac{\gamma z}{\lambda - \gamma} = 0. \qquad \text{(S 1.)}$$

13. A straight line meets two non-intersecting perpendicular straight lines in A, B. Show that, if AB is of constant length, the locus of its middle point is a circle. (S 1.)

14. ind the shortest distance between the line

$$5x - y - z = 0, \quad x - 2y + z + 3 = 0$$

and the line

$$7x - 4y - 2z = 0, \quad x - y + z - 3 = 0. \qquad \text{(S 1.)}$$

15. A chimney whose cross section is a regular hexagon rises symmetrically from a roof which slopes at an angle of 45° to the horizontal, and one diagonal of the section of the chimney lies along the ridge of the roof. Find (i) the slopes of the lines of intersection of the chimney and the roof, (ii) the angles between the roof and the faces of the chimney. (S 1.)

16. Three points, A, B, C, on a horizontal plane are chosen so that BAC is a right angle, and $AB = AC = a$. Vertical borings at A, B, C strike coal at depths h_1, h_2, h_3. Assuming that the upper surface of the coal is a plane, prove that this plane dips at an angle θ to the horizontal such that

$$a \tan \theta = \{(h_2 - h_1)^2 + (h_3 - h_1)^2\}^{\frac{1}{2}};$$

and find the line in which this plane meets the plane ABC. (S 1.)

17. Prove that the straight lines which cut two non-intersecting straight lines, so that the length intercepted is constant, are parallel to the generators of a right circular cone. (S 1.)

18. A straight line is drawn through (α, β, γ), perpendicular to each of the lines

$$(x - \alpha)/l_1 = (y - \beta)/m_1 = (z - \gamma)/n_1$$

and

$$(x - \alpha)/l_2 = (y - \beta)/m_2 = (z - \gamma)/n_2,$$

inclined to one another at an angle θ. Show that the volume of the tetrahedron formed by (α, β, γ) and the points in which the three lines cut the plane $x = 0$ is

$$\frac{\alpha^3 \sin^2 \theta}{6 l_1 l_2 (m_1 n_2 - m_2 n_1)},$$

l_1, m_1, n_1, l_2, m_2, n_2 being actual direction cosines. (S 1.)

19. A fixed point, A, is situated on the axis of x, and a fixed line OB is perpendicular to this axis at the origin, O. A variable line AP is drawn in the plane of xy, and the line of shortest distance between AP and OB passes through P. Prove that the locus of P is an ellipse which has OA for its minor axis. (S 1.)

20. Prove that the perpendicular from a fixed point $(0, 0, h)$ upon the line joining the points $(\lambda \cos \alpha, \lambda \sin \alpha, c)$, $(\lambda \cos \alpha, -\lambda \sin \alpha, -c)$ meets any plane parallel to $x = 0$ on a fixed circle, λ being a variable parameter. (S 1.)

21. A point P moves in the circle $x^2 + y^2 = r^2$, $z = 0$. Show that the straight lines drawn from P to meet two given non-intersecting straight lines, $y = x \tan \alpha$, $z = c$ and $y = -x \tan \alpha$, $z = -c$, will meet a given plane $z = h$ in points which lie on a conic. (S 1.)

22. Prove that the volume of the tetrahedron $ABCD$ is

$$\tfrac{1}{6} AB.CD.EF \sin \theta,$$

where EF is the shortest distance, and θ the angle between the edges AB and CD. (S 1.)

23. Find the equation of the circular cylinder of radius a whose axis is the line

$$(x - a)/l = (y - b)/m = (z - c)/n.$$

24. A point moves in such a way that the line joining the feet of the perpendiculars from it to two given non-intersecting straight lines subtends a right angle at a fixed point. Prove that its locus is a hyperbolic cylinder whose asymptotic planes are at right angles to the given straight lines. (C.)

25. Prove that if two straight lines are specified by their six coordinates $(l, m, n, \lambda, \mu, \nu)$ and $(l', m', n', \lambda', \mu', \nu')$, (§ 36), the condition that they are coplanar is

$$l\lambda' + m\mu' + n\nu' + l'\lambda + m'\mu + n'\nu = 0.$$

26. Verify the following construction of a straight line from its six coordinates, $(l, m, n, \lambda, \mu, \nu)$. Let k and k' be the positive values of $(l^2 + m^2 + n^2)^{-\frac{1}{2}}$ and $(\lambda^2 + \mu^2 + \nu^2)^{-\frac{1}{2}}$ respectively. Then k/k' is the length of the perpendicular, ON, drawn from the origin to the line; and the direction ON is such that OP, with direction cosines (kl, km, kn), and OQ, with direction cosines $(k'\lambda, k'\mu, k'\nu)$, and ON are a set of three lines at right angles, conformable respectively, (§ 8), with the axes of x, y and z.

CHAPTER IV

CHANGE OF AXES AND HOMOGENEOUS STRAIN

48. *Change of Axes.* Take any figure consisting of points and surfaces specified, by coordinates and equations, with regard to a given set of axes which we will call the old axes. To change the axes we have to find the coordinates of the same points, and the equations of the same surfaces, with regard to a new set of axes whose positions with regard to the old axes will be assumed to be given.

Let x, y, z denote the old coordinates of a point, and X, Y, Z the new coordinates of the same point, that is to say its coordinates with regard to the new axes. Then the problem is solved if we find, by means of the given relations between the two sets of axes, equations which give X, Y and Z in terms of x, y and z, and x, y and z in terms of X, Y and Z. These equations are called the equations of transformation for the change of axes which is contemplated.

By means of these equations, any function, $f(x, y, z)$, of the old coordinates, can be written as a function, say $F(X, Y, Z)$, of the new coordinates; so that whatever value or values $f(x, y, z)$ has at any given point, $F(X, Y, Z)$ has the same value or values at that point. A function of the coordinates of a point is sometimes referred to as a "point function". Thus $f(x, y, z)$ and $F(X, Y, Z)$ are identical point functions. And the equations $f(x, y, z) = 0$ and $F(X, Y, Z) = 0$ represent the same surface; because the points whose coordinates, referred to the old axes, satisfy the first equation, are the same as those whose coordinates, referred to the new axes, satisfy the second equation. Thus the coordinates and equations, referred to the new axes, of the points and surfaces composing the given figure, can be obtained; the notation X, Y, Z being used for current coordinates referred to the new axes.

When the new axes have been adopted, it is natural to resume the usual notation, x, y, z, for current coordinates; so that a new equation would finally be written

$$F(x, y, z) = 0.$$

Thus it is sometimes unnecessary to introduce the special notation X, Y, Z, for coordinates, except in the equations of transformation.

49. *The Origin.* The operation of a change of axes may be divided into two or more successive stages. Let us take first a change of origin, the directions of the axes remaining unchanged. Let a, b, c be the coordinates of the new origin referred to the old axes. Then the equations of transformation are

$$x = a + X, \quad y = b + Y, \quad z = c + Z.$$

Thus if α, β, γ are the coordinates of a given point referred to the old axes, its coordinates referred to the new axes are $\alpha - a$, $\beta - b$, $\gamma - c$. And if $f(x, y, z) = 0$ is the equation of a given surface referred to the old axes, its equation referred to the new axes is

$$f(a + X, \; b + Y, \; c + Z) = 0.$$

50. *Directions of Axes.* Let us now find the equations of transformation for a change of the directions of the axes, without change of origin, taking both sets of axes to be rectangular. Let (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction cosines of the new axes with reference to the old axes, so that (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3) are the direction cosines of the old axes with reference to the new axes, these being the same cosines differently grouped.

Then X is the sum of the projections, on a line with direction cosines (l_1, m_1, n_1), of the three steps, x, y, z, in the directions of the old axes. Therefore

$$X = l_1 x + m_1 y + n_1 z;$$

similarly

$$Y = l_2 x + m_2 y + n_2 z,$$

$$Z = l_3 x + m_3 y + n_3 z.$$

Also, by projecting on the old axes, we get another equivalent set of equations of transformation, namely

$$x = l_1 X + l_2 Y + l_3 Z,$$
$$y = m_1 X + m_2 Y + m_3 Z,$$
$$z = n_1 X + n_2 Y + n_3 Z.$$

Therefore $f(x, y, z)$ becomes

$$f(l_1 X + l_2 Y + l_3 Z,\ m_1 X + m_2 Y + m_3 Z,\ n_1 X + n_2 Y + n_3 Z),$$

which may be written $F(X, Y, Z)$. Thus if $f(x, y, z) = 0$ is the equation of a given surface referred to the old axes, $F(X, Y, Z) = 0$ is the equation of the same surface referred to the new axes.

The relations between the nine coefficients in these equations of transformation, which are necessary to secure that they are direction cosines of lines at right angles to one another, are

$$l_1^2 + m_1^2 + n_1^2 - 1 = 0,$$
$$l_2^2 + m_2^2 + n_2^2 - 1 = 0,$$
$$l_3^2 + m_3^2 + n_3^2 - 1 = 0,$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0,$$
$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0,$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Also by taking the other grouping of the coefficients, as sets of direction cosines, we see that the same condition can alternatively be expressed by the equations

$$l_1^2 + l_2^2 + l_3^2 - 1 = 0,$$
$$m_1^2 + m_2^2 + m_3^2 - 1 = 0,$$
$$n_1^2 + n_2^2 + n_3^2 - 1 = 0,$$
$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0,$$
$$n_1 l_1 + n_2 l_2 + n_3 l_3 = 0,$$
$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0.$$

Each of these two sets of six relations is implied by the single equation

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2,$$

which must be satisfied for every point of a figure, because its distance from the origin is unchanged. We get the first set of six relations by substituting for x, y, z their values in terms of X, Y, Z, and equating the coefficients of the corresponding terms in the two sides of this equation, which must be identical. And we get the other set of six relations, in like manner, by substituting for X, Y, Z their values in terms of x, y, z. Accordingly this single equation, regarded as valid for every point, is a complete statement of all the twelve relations; and each set of six relations implies the other six.

If the new axes of X, Y and Z are conformable with the old axes of x, y and z respectively, (§ 8), so that they can be brought into coincidence with them by a rotation about the origin, the transformation is said to be rotational. In this case, (§ 8),

$$l_1 = m_2 n_3 - m_3 n_2, \quad m_1 = n_2 l_3 - n_3 l_2, \quad n_1 = l_2 m_3 - l_3 m_2.$$

But if the two sets of axes are not conformable the transformation is irrotational. In this case

$$l_1 = -(m_2 n_3 - m_3 n_2), \quad m_1 = -(n_2 l_3 - n_3 l_2),$$
$$n_1 = -(l_2 m_3 - l_3 m_2).$$

This shows that

$$l_1 (m_2 n_3 - m_3 n_2) + m_1 (n_2 l_3 - n_3 l_2) + n_1 (l_2 m_3 - l_3 m_2),$$

which may be written

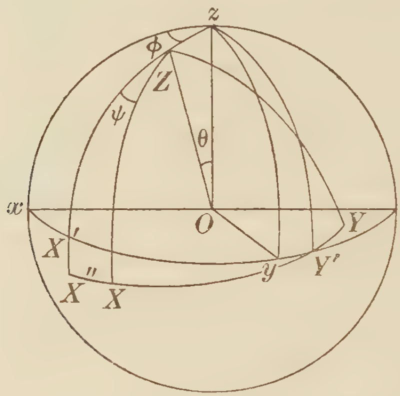
$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix},$$

is equal to 1 when the transformation is rotational, and is equal to -1 when it is irrotational.

The relations between the nine coefficients indicate that

a change of the directions of rectangular axes is specified by three independent quantities, subject to a further choice between a rotational and an irrotational change. This will now be proved.

51. *Rotational Change.* Let the new axis of Z be specified by its angular polar coordinates θ, ϕ, (§ 15). The new axis of X can then be specified by a third angle, ψ, between 0 and 2π, namely the angle of inclination of the plane of ZX to the plane of zZ. The new axis of Y is then fixed, subject only to a choice between two opposite



directions for it. In one case the transformation is rotational; in the other case it is irrotational. If this choice is immaterial, we may regard the new axes as determined by three independent quantities, θ, ϕ, ψ.

In accordance with this scheme, a rotational transformation may be effected by three successive operations, each consisting of a rotation about one of the axes. First turn the axes through an angle ϕ about the axis of z, into positions OX', OY', Oz. The equations of transformation for this operation are those of plane geometry, namely

$$x = X' \cos\phi - Y' \sin\phi, \quad y = X' \sin\phi + Y' \cos\phi, \quad z = Z',$$

X', Y', Z' being the new coordinates. Next turn the new

axes through an angle θ about the new axis of Y', into positions OX'', OY', OZ. The equations for this are similarly

$$X' = Z'' \sin\theta + X'' \cos\theta, \quad Y' = Y'',$$

$$Z' = Z'' \cos\theta - X'' \sin\theta,$$

where X'', Y'', Z'' are the new coordinates. Finally turn the axes thus obtained through an angle ψ about OZ, into positions OX, OY, OZ. The equations for this are

$$X'' = X\cos\psi - Y\sin\psi, \quad Y'' = X\sin\psi + Y\cos\psi, \quad Z'' = Z,$$

X, Y, Z being the coordinates referred to the axes in their final positions. The directions of these rotations are implied by the formulae given. They are also indicated in the diagram.

Three successive substitutions give the formulae which combine these three operations, namely

$$\begin{aligned} x = &- X(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta) \\ &\quad - Y(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta) + Z\cos\phi\sin\theta, \\ y = &\ X(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) \\ &\quad + Y(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) + Z\sin\phi\sin\theta, \\ z = &- X\cos\psi\sin\theta + Y\sin\psi\sin\theta + Z\cos\theta. \end{aligned}$$

An irrotational change would require a reversal of the direction of one axis, and this can be effected by a change of sign of one of the new coordinates.

The coefficients in these equations, which are called Euler's equations, can be verified by spherical trigonometry.

If all that is needed is to bring the axis of z into a new position proposed for it, this can be done by means of the first two rotational operations performed in succession, which give the equations

$$\begin{aligned} x &= X\cos\theta\cos\phi - Y\sin\phi + Z\sin\theta\cos\phi, \\ y &= X\cos\theta\sin\phi + Y\cos\phi + Z\sin\theta\sin\phi, \\ z &= -X\sin\theta \qquad\qquad\qquad\quad + Z\cos\theta. \end{aligned}$$

They may be derived from the former set by making ψ zero. Thus $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$ are the direction cosines of the new axis of Z with reference to the old axes; and if these direction cosines are (l, m, n), we have

$$\cos\theta = n, \quad \tan\phi = m/l.$$

Here again the coefficients in the equations of transformation might be found by spherical trigonometry.

Convenient new directions for the axes may occasionally be obvious; for example, $(1, 1, 1)$, $(1, -1, 0)$, $(1, 1, -2)$ are a set of directions at right angles.

52. *Oblique Axes*. In general any axes that are employed are rectangular, but to complete the theory, account must be taken of the case of oblique axes. Here the transformation for a change of origin is the same as for rectangular axes. And the equations for a change of the directions of the axes can be found as follows. Let l_1, l_2, l_3 be the cosines of the angles which a line in a direction perpendicular to the old plane of yz makes with the new axes, and let α be the angle which the same direction makes with the old axis of x. Then projection on this line gives

$$x\cos\alpha = l_1 X + l_2 Y + l_3 Z;$$

and we have corresponding equations for y and z. This includes the case in which one set of axes is rectangular and the other oblique.

53. Combining the operations of change of origin and change of directions of the axes, we get, in all cases, linear equations of transformation, which take the form

$$\begin{aligned} x &= a + l_1 X \ + l_2 Y \ + l_3 Z, \\ y &= b + m_1 X + m_2 Y + m_3 Z, \\ z &= c + n_1 X \ + n_2 Y \ + n_3 Z; \end{aligned}$$

and corresponding equations for X, Y, Z in terms of x, y, z. This shows that the order of an algebraical equation of a

given surface is not altered by a change of the coordinate axes, the order of such an equation being the greatest value of $\alpha + \beta + \gamma$ which occurs in any term of it when it is written in the standard form, $\Sigma A x^\alpha y^\beta z^\gamma = 0$, where α, β and γ are positive integers or zero (§ 19). It is obvious that a linear transformation cannot increase the order of such an equation; and consequently it cannot decrease it, for if it did this the reverse operation of going back to the old axes would increase it.

Therefore the surfaces represented by algebraical equations can be classified by the order of their equations. Equations of the first order represent planes. The surfaces represented by an equation of the second order,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$+ 2ux + 2vy + 2wz + d = 0,$$

also form a distinct class, and are called quadric surfaces, or quadrics.

Another deduction from the linearity of the equations of transformation is that, if $f(x, y, z)$ is a product of factors which are linear in x, y, z, the transformed function, $F(X, Y, Z)$, is a product of factors which are linear in X, Y, Z, because the substitutions of

$$a + l_1 X + l_2 Y + l_3 Z, \quad b + m_1 X + m_2 Y + m_3 Z,$$
$$c + n_1 X + n_2 Y + n_3 Z,$$

for x, y and z, can be made in each factor separately. For the same reason, if $f(x, y, z)$ is a product of identical linear factors with reference to x, y and z, the transformed function, $F(X, Y, Z)$, is a product of identical linear factors with reference to X, Y and Z.

It is convenient to apply the term "perfect square" to such a function as $\lambda(px + qy + rz)^2$, independently of whether the number λ is positive or negative; in this case the transformed function is also a perfect square.

54. *Another Interpretation of the Linear Transformation.* We have seen that the linear transformation

$$x = l_1 X + l_2 Y + l_3 Z,$$
$$y = m_1 X + m_2 Y + m_3 Z,$$
$$z = n_1 X + n_2 Y + n_3 Z,$$

in which the coefficients are restricted by the condition

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2,$$

represents the effect of a change of the directions of rectangular axes to which a given figure is referred. But another interpretation of this transformation is possible. For we might keep the coordinate axes unchanged, and consider that we are dealing with two different figures, one represented by the old coordinates and the old equations, and the other represented by the new coordinates and the new equations; each point, (x, y, z), of the first figure corresponding to a point, (X, Y, Z), of the second figure.

A rotational transformation would, according to this interpretation, give a second figure identical with the first, but rotated about the origin into a new position; for it makes no difference, so far as the equations are concerned, whether the axes are rotated with regard to the figure, or the figure with regard to the axes. And a transformation which is not rotational would give a second figure not identical, in configuration, with the first figure, but with its reflection; being the result of a rotational transformation, followed by a change of sign of one of the coordinates.

55. *Homogeneous Strain.* Let us now consider the effect of the linear transformation, interpreted in this way, when we abandon the restriction imposed on the coefficients by the condition

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2.$$

The two figures need not now be identical, nor each of them

the reflection of the other. But they have an important relation to one another.

By solving the equations of transformation, we get linear expressions for X, Y, Z in terms of x, y and z, provided that

$$l_1(m_2n_3 - m_3n_2) + m_1(n_2l_3 - n_3l_2) + n_1(l_2m_3 - l_3m_2)$$

is not zero. We shall assume that this condition is fulfilled. Then each point in either figure corresponds to a single point in the other. Each figure is derivable from the other by a change of configuration of the points composing it, combined with a shift of the figure as a whole. This change of configuration, specified by linear equations of transformation, is called a homogeneous strain.

The general theory of this strain will be discussed in a subsequent chapter, (§ 197). We shall consider here only the case in which the equations of transformation are

$$x = aX, \quad y = bY, \quad z = cZ,$$

where a, b and c are positive numbers. This transformation is the same as that which represents orthogonal projection in plane geometry. It is obvious that it is likely to be useful in the elementary geometry of surfaces. As a matter of fact the introduction of a third coordinate makes it more useful here than in plane geometry.

By reference to § 203, it will be seen that any homogeneous strain of a figure can be represented by these equations, if the coordinate axes are properly chosen, and a, b and c are not restricted, as they are here, to being positive.

56. *Corresponding Figures.* Take any figure composed of points (α, β, γ) ......, and surfaces $f(x, y, z) = 0$ .... Then the application of the transformation

$$x = aX, \quad y = bY, \quad z = cZ$$

gives a second figure composed of corresponding points $(\alpha/a, \beta/b, \gamma/c)$ ......, and surfaces $f(aX, bY, cZ) = 0$ ......, the notation X, Y, Z being employed for current coordinates in the second figure. If either figure is known

the other one can be constructed from it; and certain properties of either figure are derivable from those of the other.

Each point of either figure corresponds to a single point of the other. Contiguous points in one figure correspond to contiguous points in the other, and points at infinity to points at infinity. A plane,

$$Ax + By + Cz + D = 0,$$

in the first figure, corresponds to a plane

$$AaX + BbY + CcZ + D = 0$$

in the second figure. This shows that a set of parallel planes in one figure corresponds to a set of parallel planes in the other. A straight line in one figure corresponds to a straight line in the other, because it is the intersection of two planes. And two parallel straight lines in one figure correspond to two parallel straight lines in the other, because they are in one plane and do not intersect. A straight line,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

drawn through the point (α, β, γ), with direction cosines (l, m, n), becomes in the second figure the straight line

$$\frac{aX-\alpha}{l} = \frac{bY-\beta}{m} = \frac{cZ-\gamma}{n};$$

that is to say a line through the point $(\alpha/a, \beta/b, \gamma/c)$, with direction cosines proportional to l/a, m/b, n/c.

Let r be the positive length of a line in the first figure, drawn from a point (α, β, γ) to a point $(\alpha', \beta', \gamma')$, with direction cosines (l, m, n); and R the positive length of the corresponding line drawn, in the second figure, from the point $(\alpha/a, \beta/b, \gamma/c)$ to the point $(\alpha'/a, \beta'/b, \gamma'/c)$, with direction cosines (λ, μ, ν). Then

$$\alpha' = \alpha + lr, \quad \beta' = \beta + mr, \quad \gamma' = \gamma + nr,$$

and $$\frac{\alpha'}{a} = \frac{\alpha}{a} + \lambda R, \quad \frac{\beta'}{b} = \frac{\beta}{b} + \mu R, \quad \frac{\gamma'}{c} = \frac{\gamma}{c} + \nu R;$$

therefore $$\frac{R}{r} = \frac{l}{a\lambda} = \frac{m}{b\mu} = \frac{n}{c\nu}.$$

Therefore the three ratios $l : \lambda$, $m : \mu$ and $n : \nu$ are all positive. Also

$$\frac{R^2}{r^2} = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2},$$

and

$$\frac{r^2}{R^2} = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2.$$

Thus the ratio of a length in one figure to the corresponding length in the other depends only on its direction; and the ratios to one another of lengths in one direction, whether parallel lengths or segments of one line, are the same as those of the corresponding lengths in the other figure.

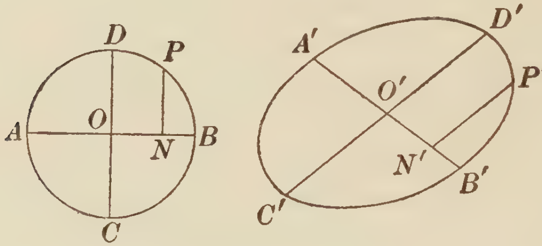
The coordinate axes in one figure correspond to the coordinate axes in the other. A tangent line at a point of a surface in one figure corresponds to a tangent line at the corresponding point of the corresponding surface, because it is either the limiting case of a chord of a plane section of the surface, or else a straight line on the surface. Also the tangent plane at a point of a surface corresponds to the tangent plane at the corresponding point of the corresponding surface, being the plane which contains all the tangent lines at the point in question.

The ratio of any volume in the first figure to the corresponding volume in the second figure is $abc : 1$. This can be inferred from the integral calculus formula for a volume, either $\iiint dx\,dy\,dz$ or $\iint z\,dx\,dy$.

Similarly the ratio to one another of two areas in a given plane, in the first figure, is the same as the ratio to one another of the corresponding areas, in the corresponding plane, in the second figure; because the two areas can be divided into corresponding elements. It should be noticed that any repeated pattern, (such as that of a wall-paper), drawn in one plane, obviously corresponds to a repeated pattern drawn in the other plane.

57. *Circle and corresponding Ellipse.* Any circle in the first figure corresponds to an ellipse in the second figure. To prove this, let AOB, COD be any pair of diameters of the circle at right angles to one another; and draw any ordinate of the circle, PN, parallel to COD. Then

$$\left(\frac{ON}{OB}\right)^2 + \left(\frac{NP}{OD}\right)^2 = 1.$$



In the second figure we have the corresponding curve in one plane, but in general not in the plane of the circle; and we have corresponding lines $A'O'B'$, $C'O'D'$, bisected at O', and an ordinate $P'N'$ parallel to $C'O'D'$. And as the ratios to one another of parallel lengths are unchanged,

$$\frac{O'N'}{O'B'} = \frac{ON}{OB} \quad \text{and} \quad \frac{N'P'}{O'D'} = \frac{NP}{OD}.$$

Therefore
$$\left(\frac{O'N'}{O'B'}\right)^2 + \left(\frac{N'P'}{O'D'}\right)^2 = 1.$$

This shows that the curve is an ellipse of which $A'O'B'$ and $C'O'D'$ are a pair of conjugate diameters.

Take any two circles in the same plane, or in parallel planes, in the first figure. Corresponding to them we have in the second figure two ellipses, similar and similarly placed. And as the two circles are converted into ellipses in the same way, differing only with regard to the scale on which they are drawn, the ratio of the areas of the two ellipses is the same as the ratio of the areas of the corresponding circles.

58. *Examples.*

1. Find the equations of transformation for a change of the directions of the axes when the old axes are rectangular, and the new axes are oblique with direction cosines (l, m, n), (l', m', n'), (l'', m'', n'').

2. Show that the condition that two straight lines through the origin can be converted into straight lines at right angles by the transformation $x = aX$, $y = bY$, $z = cZ$, is that the two lines are not in the same pair of opposite octants of the axes.

3. Find the equation of the surface into which the sphere

$$x^2 + y^2 + z^2 = 1$$

is converted by the general linear transformation

$$x = lX + mY + nZ,$$
$$y = l'X + m'Y + n'Z,$$
$$z = l''X + m''Y + n''Z.$$

And show that it provides a proof that any linear transformation converts any plane section of a sphere into an ellipse.

CHAPTER V

THE SPHERE

59. The quadric surface represented by the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is called a sphere. It will be noticed that a change of coordinate axes does not alter the terms of the second order in this equation. By writing the equation in the form

$$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d,$$

it is seen that it has a real locus only when $u^2 + v^2 + w^2 - d$ is positive or zero.

Dealing here only with the real locus, the equation may be written

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = a^2,$$

where a is positive or zero. And as the left-hand side of this equation is the square of the distance between the points (x, y, z) and (α, β, γ), this shows that the real locus is a sphere in the ordinary sense of that term, with centre (α, β, γ) and radius of length a; unless a is zero, in which case it is a single point, which may be regarded as a sphere of zero radius.

The coordinates of the centre are $(-u, -v, -w)$, and the square of the radius is $u^2 + v^2 + w^2 - d$.

A single sphere can in general be drawn through four given points; for if the coordinates of these points are substituted for x, y, z in the general equation of a sphere, we have four linear equations for u, v, w and d.

60. *The Symmetry of a Sphere.* It will be assumed here that the following properties of a sphere, which are immediate deductions from the consideration of symmetry, are known.

All diameters are at right angles to the tangent planes at their extremities. All plane sections are circles. The centres of sections by a set of parallel planes are on the diameter which is at right angles to these planes. A diametral plane, that is to say a plane through the centre, bisects all chords at right angles to it. Any set of three diameters, at right angles to one another, are such that the plane drawn through any two of them bisects all chords parallel to the third.

61. *Equation of Tangent Plane.* Taking the centre of the sphere for the origin, O, the equation of the surface is

$$x^2 + y^2 + z^2 = a^2.$$

Let P, (x', y', z'), be a point on the surface; then the equation of the tangent plane at P is

$$x'x + y'y + z'z = a^2,$$

for this is the equation of a plane drawn through the point (x', y', z'), and at right angles to the direction (x', y', z').

62. *Polar Plane.* Let $P, (x', y', z')$, be any point. Then the plane

$$x'x + y'y + z'z = a^2$$

is called the polar plane of P with regard to the sphere, and the point P is called the pole of this plane. Let r be the length OP, and k the length of the perpendicular from O on the polar plane. The equation of the plane is rectified, (§ 30), by dividing by r; therefore $k = a^2/r$. This shows that the relation between a point and its polar plane is independent of the choice of coordinate axes. It will be noticed that the polar plane of P is at right angles to OP, and that it is the tangent plane at P when P is on the surface, and that it passes to infinity when P approaches the centre.

The condition that a point $Q, (x'', y'', z'')$, is on the polar plane of P is

$$x'x'' + y'y'' + z'z'' = a^2.$$

And the symmetry of this equation shows that if any point Q is on the polar plane of a point P, P is on the polar plane of Q. Through any point P, outside the sphere, draw tangent planes to the sphere. Each of these planes is the polar plane of its point of contact, therefore these points of contact all lie on the polar plane of P. Therefore if a cone is drawn, with vertex P, enveloping the sphere, its points of contact with the sphere are in one plane. This is the polar plane of P, and may also be called the plane of contact of the cone. This cone is the surface generated by tangent lines of the sphere drawn through P.

Through a given point P, (x', y', z'), draw a straight

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line

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho$$

with direction cosines (l, m, n). And let ρ_1, ρ_2 be the values of ρ for the points in which it meets the sphere, and ρ_3 the value of ρ for the point in which it meets the polar plane of P. Then ρ_1, ρ_2 are the roots of the quadratic equation

$$(x' + l\rho)^2 + (y' + m\rho)^2 + (z' + n\rho)^2 = a^2,$$

or $$\rho^2 + 2\rho\,(lx' + my' + nz') + x'^2 + y'^2 + z'^2 - a^2 = 0;$$

and ρ_3 is given by the equation

$$x'\,(x' + l\rho_3) + y'\,(y' + m\rho_3) + z'\,(z' + n\rho_3) = a^2.$$

Therefore $$\rho_1 + \rho_2 = -\,2\,(lx' + my' + nz'),$$

and $$\rho_1\rho_2 = x'^2 + y'^2 + z'^2 - a^2,$$

and $$\rho_3 = -\frac{x'^2 + y'^2 + z^2 - a^2}{lx' + my' + nz'}.$$

Therefore $$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{2}{\rho_3}.$$

The product $\rho_1\rho_2$ is called the power of the point P with regard to the sphere. It is shown here that it depends only on the distance of P from the centre, and that it is positive or negative or zero according as P is outside, or inside, or on the sphere.

The quadratic equation for ρ, written above, shows how the equation of the tangent plane at a point P, (x', y', z'), can be derived from the equation of the surface. For if P is on the surface the equation for ρ is

$$\rho^2 + 2\rho\,(lx' + my' + nz') = 0,$$

therefore the condition that the line through P is a tangent line is $lx' + my' + nz' = 0$; that is to say the tangent lines are the lines through P at right angles to the radius OP.

63. If the equation of a sphere is given in the form

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = a^2,$$

the centre being any given point, (α, β, γ), the equation of the polar plane of a point (x', y', z'), which is the tangent plane when the point (x', y', z') is on the surface, is

$$(x' - \alpha)(x - \alpha) + (y' - \beta)(y - \beta) + (z' - \gamma)(z - \gamma) = a^2,$$

because this agrees with the former equation when the origin is shifted to the point (α, β, γ). This equation may be written

$$x'x + y'y + z'z - \alpha\,(x + x') - \beta\,(y + y') - \gamma\,(z + z') \\ + \alpha^2 + \beta^2 + \gamma^2 - a^2 = 0.$$

Thus if the equation of the sphere is written

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

the equation of the polar plane of (x', y', z'), or tangent plane when (x', y', z') is on the surface, is

$$x'x + y'y + z'z + u\,(x + x') + v\,(y + y') + w\,(z + z') + d = 0,$$

and the equation of the polar plane of the origin is

$$ux + vy + wz + d = 0.$$

For the specification of a polar plane we still have the formula $1/\rho_1 + 1/\rho_2 = 2/\rho_3$, (independent of the axes). That is to say, any chord is divided harmonically by any point on it and the polar plane of that point.

A plane $Ax + By + Cz + D = 0$ is a tangent plane of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

if, and not unless, the length of the perpendicular from the centre on the plane is equal to the radius of the sphere. Now the coordinates of the centre are $-u$, $-v$, $-w$, and the square of the radius is $u^2 + v^2 + w^2 - d$. Therefore the condition that the plane is a tangent plane is

$$(Au + Bv + Cw - D)^2 = (A^2 + B^2 + C^2)(u^2 + v^2 + w^2 - d).$$

This relation between A, B, C, D is called the tangential equation of the sphere.

64. *Enveloping Cone.* The tangent lines of the sphere, $x^2 + y^2 + z^2 = a^2$, drawn through a given external point P, (x', y', z'), are the generators of a cone with vertex P. To find the equation of this cone, let Q, (x, y, z), be any point on it. Then the coordinates of a point in which the line PQ meets the sphere may be written, (§ 13),

$$\frac{\lambda x + \mu x'}{\lambda + \mu}, \quad \frac{\lambda y + \mu y'}{\lambda + \mu}, \quad \frac{\lambda z + \mu z'}{\lambda + \mu},$$

the ratio λ/μ being given by the equation

$$(\lambda x + \mu x')^2 + (\lambda y + \mu y')^2 + (\lambda z + \mu z')^2 = (\lambda + \mu)^2 a^2;$$

or $$(x^2 + y^2 + z^2 - a^2)\lambda^2 + 2(xx' + yy' + zz' - a^2)\lambda\mu + (x'^2 + y'^2 + z'^2 - a^2)\mu^2 = 0.$$

And the condition that the line is a tangent line is that this quadratic equation for λ/μ has equal roots, that is to say

$$(x'^2 + y'^2 + z'^2 - a^2)(x^2 + y^2 + z^2 - a^2) - (x'x + y'y + z'z - a^2)^2 = 0.$$

Therefore this is the equation of the cone.

65. *Polar Lines.* Let P, Q be any two points, whose polar planes intersect in a line RS. Then the polar planes of the points R and S pass through P and through Q; and they are not coincident, therefore they intersect in the line PQ. Thus any given straight line, PQ, which does not pass through the centre of the sphere, defines another straight line, RS; the relation between these lines being that the polar plane of any point on either of them passes through the other. Each line is called the polar line of the other with regard to the sphere. The two lines are at right angles to one another, for one of them must cut the sphere, and the other is the line of intersection of the tangent planes at the two points of section. That all pairs of polar lines are at right angles is a special property of a sphere. In the case of the corresponding proposition for other quadric surfaces, polar lines are not in general at right angles.

66. *Polar Reciprocals.* Let a plane be conceived to move continuously so as to be a tangent plane of a given surface. Then the pole of this plane, with respect to a given sphere, will move continuously so that it traces out another surface, or possibly only a curve.

Let us first suppose that it traces out another surface. Two positions of the plane define a straight line, namely their line of intersection, which in the limit, when they tend to coincidence, is a tangent line of the first surface; and the straight line through the corresponding poles is, in the limit, a tangent line of the second surface. Each of these tangent lines is the polar line of the other with respect to the sphere. And the relation between them is reciprocal. Thus the poles of tangent planes of the second surface are on the first one. And if either surface is given, the other can be constructed from it. Each is called the polar reciprocal of the other with respect to the sphere. And from any proposition involving given points and planes related to one surface, a proposition can be deduced involv-

ing planes and points related to the other surface, namely the planes which are the polar planes of the given points, and the points which are the poles of the given planes. This procedure of deriving one proposition from another is called reciprocation.

The case in which the pole of a moving plane traces out a curve, instead of a surface, is that in which the plane is the tangent plane of a developable surface. In fact a simple definition of a developable surface is that it is the polar reciprocal of a curve. A point moving along a curve moves with only one degree of freedom, and a developable is a surface whose tangent plane has only one degree of freedom. The generating lines of the developable are the polar lines of the tangents of the curve. If the curve is a plane curve, all the tangent planes of the developable pass through a point, namely the pole of the plane of the curve, so that the developable is a cone; except when the plane of the curve passes through the centre of the sphere, in which case the developable is a cylinder, this being the limiting case of a cone when its vertex passes to infinity.

It will be found that all quadric surfaces have pole and polar properties similar to those of a sphere. But for the purpose of reciprocation it is usual to employ a sphere.

67. *Radical Plane.* Let us write S_1 for

$$(x - \alpha_1)^2 + (y - \beta_1)^2 + (z - \gamma_1)^2 - a_1^2$$

and S_2 for

$$(x - \alpha_2)^2 + (y - \beta_2)^2 + (z - \gamma_2)^2 - a_2^2,$$

and so on. Then

$$S_1 = 0, \quad S_2 = 0 \;\ldots\ldots$$

are the equations of a system of spheres. When (x, y, z) is a point outside the sphere $S_1 = 0$, S_1 is equal to the square of the length of a tangent line drawn from the point (x, y, z) to this sphere; and by reference to § 62,

it will be seen that it is, in all cases, the power of the point (x, y, z) with regard to the sphere. Now

$$S_1 - S_2 = 0$$

is the equation of a plane, for it is an equation of the first order. It is called the radical plane of the pair of spheres $S_1 = 0$, $S_2 = 0$. The equation shows that if these spheres intersect, the radical plane passes through all points of their intersection; also that, in all cases, tangent lines drawn from a point on this plane to the two spheres are equal in length.

If k is not unity, $S_1 - kS_2 = 0$ is the equation of a sphere; and therefore represents a sphere drawn through the circle in which the spheres $S_1 = 0$ and $S_2 = 0$ meet, if they meet. And whether they meet or not, it represents a sphere such that, if tangents are drawn from a point on it to the spheres $S_1 = 0$ and $S_2 = 0$, the ratio of their lengths is independent of the position of this point on the sphere.

The form of the equation of a radical plane shows that the three radical planes of three spheres, taken in pairs, have a common line of intersection if they are not parallel; and that, in general, the six radical planes of four spheres, taken in pairs, meet in a point. This point may be called the radical centre of the four spheres.

68. *Examples.*

1. Find the coordinates of the centre, and the radius, of the sphere

$$x^2 + y^2 + z^2 - 2x + 3y - 5z = 4.$$

2. Find the locus of a point whose distances from two given points are in a given ratio.

3. A sphere of radius r touches the three coordinate planes; find the equations of the circle in which the sphere is cut by the plane $z = \frac{1}{2}r$. Also find the equations of the planes which touch the sphere and which have their intercepts on the axes Ox, Oy, Oz respectively in the ratio $2 : 3 : 5$. (S 1.)

4. Find separately the equations of the two planes which pass through the straight line $x = y = z$, and touch the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 7 = 0,$$

and show that they are at right angles. (S 1.)

5. Prove that a sphere can be drawn to pass through the mid-points of the edges of the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$, $x/a + y/b + z/c = 2$.

$$[(x - \tfrac{1}{2}a)^2 + (y - \tfrac{1}{2}b)^2 + (z - \tfrac{1}{2}c)^2 = \tfrac{1}{4}(a^2 + b^2 + c^2).] \qquad \text{(S 1.)}$$

6. A point K is at a constant distance $2a$ from the origin, and points P, Q, R are taken on the axes in such a way that KP, KQ, KR are mutually perpendicular. Prove that the plane PQR always touches a fixed sphere of radius a. (S 1.)

7. Show that the general equation of all spheres through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is

$$x^2 + y^2 + z^2 - ax - by - cz - \lambda(x/a + y/b + z/c - 1) = 0.$$

Find the value of λ so that this sphere may cut orthogonally the sphere whose equation is

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0. \qquad \text{(S 1.)}$$

8. Two circles in different planes meet in the two points A, B; show that a sphere, S, can be drawn through both circles. If the circles intersect at right angles, prove that the plane of either circle passes through the pole of the plane of the other one with respect to the sphere S. (S 1.)

9. Prove that the plane $x + 2y - z = 4$ cuts the sphere

$$x^2 + y^2 + z^2 - x + z - 2 = 0$$

in a circle of radius unity, and find the equation of the sphere which has this circle for one of its great circles, see § 178. (S 1.)

10. Find the necessary and sufficient conditions that a sphere can be drawn to pass through the circles

$$x = 0, \quad y^2 + z^2 + 2g_1y + 2h_1z + c_1 = 0;$$

$$y = 0, \quad z^2 + x^2 + 2h_2z + 2f_2x + c_2 = 0;$$

$$z = 0, \quad x^2 + y^2 + 2f_3x + 2g_3y + c_3 = 0.$$

If they are satisfied, find the equation of the projection on the plane $x = 0$ of the intersection of the sphere with the plane

$$lx + my + nz = p. \qquad \text{(S 1.)}$$

11. Find an expression for the angle of intersection of two spheres which are given by their equations in rectangular Cartesian coordinates. Prove that the locus of the centre of a sphere which cuts three given spheres at the same angle is a plane. (S 1.)

12. Three spheres, of radii r_1, r_2, r_3, have their centres, A, B, C, at the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, and $r_1^2 + r_2^2 + r_3^2$ is equal to $a^2 + b^2 + c^2$. A fourth sphere passes through the origin and A, B, C. Show that the radical centre of the four spheres lies on the plane

$$ax + by + cz = 0. \qquad \text{(S 1.)}$$

13. Find the equation of the sphere through the origin O and three points A, B, C whose coordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. And show that if O' is the centre of this sphere, the sphere on OO' as diameter passes through the middle points of the six edges of the tetrahedron $OABC$, and through the feet of the perpendiculars from O on the sides of the triangle ABC. (S 1.)

14. Find the condition that the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

may touch the line $(x - a)/l = (y - b)/m = (z - c)/n$. Show that this line is touched by two spheres, each of which passes through the points $(0, 0, 0)$, $(2a, 0, 0)$, $(0, 2b, 0)$. Show further that, if l, m, n are direction cosines, the distance between the centres of the spheres is

$$2n^{-2}\{c^2 - n^2(a^2 + b^2 + c^2)\}^{\frac{1}{2}}. \qquad \text{(S 1.)}$$

15. Show that, if $af + bg + ch = 0$, any sphere drawn through the origin and through the points $(\lambda a, 0, 0)$, $(0, \lambda b, 0)$, $(0, 0, \lambda c)$ is cut orthogonally by the sphere

$$x^2 + y^2 + z^2 + 2fx + 2gy + 2hz = 0. \qquad \text{(S 1.)}$$

16. Find the equation of the sphere which passes through the point $(2, 3, 6)$ and the feet of the perpendiculars from this point on the coordinate planes. Also find the equations of the tangent planes of this sphere which are parallel to the plane $2x + 2y + z = 1$, and the coordinates of their points of contact. (S 1.)

17. A sphere moves in contact with two fixed straight lines which intersect at an angle 2α. Show that the centre of the sphere describes an ellipse of eccentricity equal to $\cos\alpha$. If in any position the planes which touch the sphere at the points of contact make an angle θ with the plane of the fixed straight lines, and if the line of intersection of these planes make an angle ϕ with the same fixed plane, show that

$$\tan\phi = \tan\theta \sin\alpha. \qquad \text{(S 1.)}$$

CHAPTER VI

THE ELLIPSOID

69. A surface whose equation can, by making suitable choice of coordinate axes, be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is called an ellipsoid. Thus a sphere is a particular case of an ellipsoid. The numbers a, b, c will be taken to be positive.

It is clear that in adopting, as we shall now do, this form of the equation of the surface, referred to rectangular axes, we are taking axes very symmetrically placed with regard to it. If a, b and c are unequal, almost any change of axes would disturb this form, and might produce an equation containing as many as ten terms (§ 19); in fact the only change that would not disturb it is a reversal of the direction of one or more of the axes.

The point here taken as origin is called the centre of the ellipsoid; the lines taken as coordinate axes are called the axes, or principal axes of the ellipsoid; and the planes taken as coordinate planes are called the principal planes. Chords through the centre are called diameters, and planes through the centre are called diametral planes.

The equation shows that the sections of the surface by the principal planes are ellipses; and that the sections by planes parallel to any one of the principal planes are similar and similarly situated ellipses. For example, the section by the plane $z=k$, where $k^2<c^2$, is the ellipse

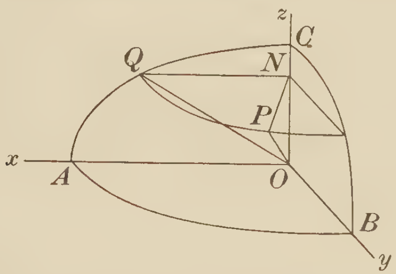
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}, \quad z=k.$$

When $k^2>c^2$ this plane does not meet the surface. From these sections the surface can be constructed. The portion of it in one octant is shown in the diagram, and

the portions in the other octants are the same as this or reflections of it. This symmetry shows that all diameters are bisected at the centre, and that every chord at right angles to a principal plane is bisected by it.

It is obvious that the lengths of the diameters which are the axes of the surface are $2a$, $2b$, $2c$. It will be assumed here, in general, that a, b and c are unequal, and $a > b > c$.

Through any point P on the ellipsoid draw the section by a plane parallel to the plane of xy, meeting the plane of zx in Q, as shown in the diagram; and let N be the



projection of P on the axis of z. Then QN is the semi-major axis of this section, because it is similar to the section by the plane of xy. Therefore $QN > PN$. Also

$$OP^2 = ON^2 + PN^2, \text{ and } OQ^2 = ON^2 + QN^2;$$

therefore $OP < OQ$. And $OQ < a$; therefore $OP < a$. Similarly it can be proved that $OP > c$. Therefore the lengths of all diameters are between $2a$ and $2c$.

70. *Sphere corresponding to the Ellipsoid.* The most obvious general property of an ellipsoid is its relation to a sphere. This is seen by making the transformation

$$x = aX, \quad y = bY, \quad z = cZ.$$

Taking as a first figure, (§ 56), the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and planes, lines and points associated with it; we get in

this way a corresponding second figure, with coordinate axes of X, Y, Z, and the sphere

$$X^2 + Y^2 + Z^2 = 1,$$

corresponding to the ellipsoid, and the planes, lines and points which correspond to those drawn in the first figure. And the relations between the two figures are those stated in a previous chapter, (§ 56).

A point (x', y', z') in the first figure corresponds to a point $(x'/a, y'/b, z'/c)$ in the second figure; and a direction (l, m, n) in the first figure corresponds to a direction $(l/a, m/b, n/c)$ in the second figure.

The centre of the ellipsoid corresponds to the centre of the sphere. A diameter of the ellipsoid, and chords parallel to it, correspond to a diameter of the sphere, and chords parallel to it. Diametral planes of the ellipsoid correspond to diametral planes of the sphere, and each of them bisects a series of parallel chords.

All plane sections of the sphere are circles, therefore all plane sections of the ellipsoid are ellipses, (§ 57). Take the sections of the sphere by any given set of parallel planes. They are circles with their centres on a diameter of the sphere. Corresponding to these circles we have, in the first figure, ellipses similar to one another, and similarly placed, which are the sections of the ellipsoid by the corresponding set of parallel planes, and have their centres on a straight line which is a diameter of the ellipsoid.

Draw any number of spheres in the second figure. We know that the ratio of a length in one figure to the corresponding length in the other figure depends only on its direction. Therefore these spheres correspond to a system of similar and similarly situated ellipsoids in the first figure.

71. *Equation of Tangent Plane.* The tangent plane of the ellipsoid at a point (x', y', z') corresponds to the tangent plane of the sphere at the corresponding point, $(x'/a, y'/b, z'/c)$.

Now the equation of the latter plane has been found, (§ 61). It is

$$\frac{x'}{a} X + \frac{y'}{b} Y + \frac{z'}{c} Z = 1,$$

and the equation of the corresponding plane in the first figure is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} + \frac{z'z}{c^2} = 1,$$

(§ 56); therefore this is the equation of the tangent plane of the ellipsoid at the point (x', y', z').

Let p be the length, and (λ, μ, ν) the direction cosines, of the perpendicular drawn from the origin to this plane. Then the equation of the plane is

$$\lambda x + \mu y + \nu z = p,$$

where $$\lambda = p\frac{x'}{a^2}, \quad \mu = p\frac{y'}{b^2}, \quad \nu = p\frac{z'}{c^2};$$

therefore $$\frac{1}{p^2} = \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}.$$

Also $$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1,$$

therefore $$p^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2;$$

accordingly the equation of the tangent plane is

$$\lambda x + \mu y + \nu z = \sqrt{(a^2\lambda^2 + b^2\mu^2 + c^2\nu^2)},$$

where the square root has its positive value.

Let (l, m, n) be the direction cosines of the radius, of length r, drawn from the centre to the point (x', y', z'), and θ the angle between the directions of this radius and of the perpendicular, p. Then

$$x' = lr, \quad y' = mr, \quad z' = nr,$$

and $$\frac{1}{r^2} = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}.$$

This is the equation of the surface in terms of the polar coordinates r, l, m, n.

Also $$\lambda = pr\frac{l}{a^2}, \quad \mu = pr\frac{m}{b^2}, \quad \nu = pr\frac{n}{c^2};$$

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therefore the ratios λ/l, μ/m, ν/n are positive; and the radius coincides with the perpendicular only when pr is equal to either a^2 or b^2 or c^2. If a, b and c are unequal, this requires that the direction cosines of the radius should be either $(1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$; so that the principal axes are the only diameters which are at right angles to the tangent planes at their extremities.

72. *Enveloping Cone.* The enveloping cone of the ellipsoid corresponds to the enveloping cone, (§ 64), of the sphere. Thus its equation can be written down; or can be found independently by the same procedure. It is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) - \left(\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}-1\right)^2=0,$$

the point (x', y', z') being the vertex.

73. *Normal.* The equation of the tangent plane at P, (x', y', z'), gives the equations of the normal at the same point, namely

$$\frac{a^2}{x'}(x-x')=\frac{b^2}{y'}(y-y')=\frac{c^2}{z'}(z-z');$$

or in the parametric form of § 12,

$$x=x'+\frac{px'}{a^2}\rho,\quad y=y'+\frac{py'}{b^2}\rho,\quad z=z'+\frac{pz'}{c^2}\rho,$$

the parameter, ρ, being positive in the direction from the inside of the ellipsoid towards the tangent plane.

If P is in one of the principal planes, the normal at P is in that plane, either x' or y' or z' being zero. If P is not in a principal plane, let F, G, H be the points in which the normal cuts the planes $x=0$, $y=0$, $z=0$ respectively. Then the parametric equations of the normal give

$$PF=\frac{a^2}{p},\quad PG=\frac{b^2}{p},\quad PH=\frac{c^2}{p},$$

ρ being in each case negative. Thus, on the assumption

that $a > b > c$, the points P, H, G, F always occur in this order.

By means of the same equations, the question of the number of normals that can be drawn to the surface from any given point, (α, β, γ), can be investigated. Let (x', y', z') be the coordinates of the point on the surface to which one of these normals is drawn. Then the equations connecting α, β, γ with x', y', z' may be written

$$\alpha = \left(1 + \frac{\sigma}{a^2}\right) x', \quad \beta = \left(1 + \frac{\sigma}{b^2}\right) y', \quad \gamma = \left(1 + \frac{\sigma}{c^2}\right) z';$$

also $$x'^2/a^2 + y'^2/b^2 + z'^2/c^2 = 1.$$

Therefore $$\frac{a^2\alpha^2}{(a^2+\sigma)^2} + \frac{b^2\beta^2}{(b^2+\sigma)^2} + \frac{c^2\gamma^2}{(c^2+\sigma)^2} = 1.$$

This is an equation for σ of the sixth degree, and each real value of σ which satisfies it gives one of the real normals that can be drawn to the surface from the point (α, β, γ).

74. *Conjugate Diameters.* Three diameters of the ellipsoid, such that the plane through any two of them bisects all chords parallel to the third, are called a set of conjugate diameters.

This definition shows that sets of conjugate diameters of the ellipsoid correspond to sets of diameters of the sphere with the same property. But sets of diameters of the sphere defined by this property are well known. They are the sets of three diameters at right angles to one another. And there are any number of such sets; and any given diameter of the sphere is a member of any number of such sets. Therefore there are any number of sets of conjugate diameters of the ellipsoid. And any given diameter of the ellipsoid is a member of any number of sets of conjugate diameters.

A diameter of the sphere being given, the plane through the centre, at right angles to it, contains all diameters at

right angles to it, and bisects all chords parallel to it. Therefore, for the ellipsoid, if one diameter, of a set of conjugate diameters, is given, the other two are in a certain plane which bisects all chords parallel to it. This plane and the given diameter are said to be conjugate to one another; and the plane is also called the diametral plane of the chords which it bisects.

A diameter of the sphere is at right angles to the tangent planes at its extremities; therefore the tangent planes of the ellipsoid at the extremities of a diameter are parallel to the diametral plane which is conjugate to that diameter.

Any two members of a set of conjugate diameters of the ellipsoid are conjugate diameters of the ellipse which is the section of the surface by the plane of these diameters, because each diameter must bisect chords in this plane drawn parallel to the other one.

If, as has been assumed, a, b and c are unequal, there is only one set of conjugate diameters of the ellipsoid at right angles to one another, namely the principal axes, because no other diameters are at right angles to the tangent planes at their extremities, (§ 71).

Every set of three radii of the sphere at right angles to one another gives a corresponding set of conjugate radii, (that is to say conjugate semi-diameters), of the ellipsoid.

Let a_1, a_2, a_3 be the lengths, and (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) the direction cosines, of a set of conjugate central radii of the ellipsoid; and let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the coordinates of their extremities. Then the direction cosines of the three corresponding radii of the sphere may be written in either of the two forms

$$\left(\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}\right), \quad \left(\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}\right), \quad \left(\frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}\right);$$

and

$$\left(\frac{a_1 l_1}{a}, \frac{a_1 m_1}{b}, \frac{a_1 n_1}{c}\right), \quad \left(\frac{a_2 l_2}{a}, \frac{a_2 m_2}{b}, \frac{a_2 n_2}{c}\right), \quad \left(\frac{a_3 l_3}{a}, \frac{a_3 m_3}{b}, \frac{a_3 n_3}{c}\right);$$

whichever may be most convenient. And the fact that they are at right angles is expressed by the equations

$$\frac{l_2 l_3}{a^2} + \frac{m_2 m_3}{b^2} + \frac{n_2 n_3}{c^2} = 0,$$

$$\frac{l_3 l_1}{a^2} + \frac{m_3 m_1}{b^2} + \frac{n_3 n_1}{c^2} = 0,$$

$$\frac{l_1 l_2}{a^2} + \frac{m_1 m_2}{b^2} + \frac{n_1 n_2}{c^2} = 0.$$

Therefore this is the condition for a set of diameters of the ellipsoid being conjugate.

The direction cosines of the radii of the sphere also satisfy all the rest of the numerous relations between the direction cosines of three directions at right angles which are given in § 50. Several of these deserve notice.

We have of course

$$\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2} = \frac{1}{a_1^2},$$

and two similar equations.

Also
$$a_1^2 l_1^2 + a_2^2 l_2^2 + a_3^2 l_3^2 = a^2,$$

$$a_1^2 m_1^2 + a_2^2 m_2^2 + a_3^2 m_3^2 = b^2,$$

$$a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2 = c^2.$$

Whence we get by addition

$$a_1^2 + a_2^2 + a_3^2 = a^2 + b^2 + c^2.$$

Let r_1, r_2 be the lengths of the semi-axes of the ellipse which is the section of the ellipsoid by the diametral plane conjugate to a_1. Then we know from conics that $a_2^2 + a_3^2$ is equal to $r_1^2 + r_2^2$; therefore

$$r_1^2 + r_2^2 = a^2 + b^2 + c^2 - a_1^2.$$

Also
$$x_1^2 + x_2^2 + x_3^2 = a^2,$$

$$y_1^2 + y_2^2 + y_3^2 = b^2,$$

$$z_1^2 + z_2^2 + z_3^2 = c^2.$$

Also
$$y_1z_1 + y_2z_2 + y_3z_3 = 0,$$
$$z_1x_1 + z_2x_2 + z_3x_3 = 0,$$
$$x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

Also, as in § 50,

$$\pm\frac{x_1}{a} = \frac{1}{bc}(y_2z_3 - y_3z_2), \quad \pm\frac{y_1}{b} = \frac{1}{ca}(z_2x_3 - z_3x_2),$$
$$\pm\frac{z_1}{c} = \frac{1}{ab}(x_2y_3 - x_3y_2);$$

and
$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \pm\, abc;$$

the ambiguous signs being all +, or all −, according as the radii, 1, 2, 3, of the sphere, are, or are not, conformable, (§ 8), with the axes of x, y and z.

75. We know, (§ 56), that the ratio of any volume in the first figure to the corresponding volume in the second figure is abc. The volume of a sphere of unit radius is $\frac{4}{3}\pi$; therefore the volume of the ellipsoid is $\frac{4}{3}\pi abc$. The volume of a cube in which a sphere of unit radius can be inscribed is 8; therefore the volume of any parallelepiped formed by tangent planes of the ellipsoid at the extremities of three conjugate diameters is $8abc$. Now if r_1, r_2 are the semi-axes of the central section parallel to one face of the parallelepiped, we know from conics that the area of this face is $4r_1r_2$. Therefore if p is the length of the perpendicular from the centre on this face, the volume of the parallelepiped is $8r_1r_2p$. Therefore

$$r_1r_2 = abc/p.$$

From the expressions for $r_1^2 + r_2^2$ and r_1r_2, the values of r_1 and r_2 can be calculated.

76. *Equation referred to Conjugate Diameters.* Let (ξ, η, ζ) be the coordinates of a point on the ellipsoid referred to a set of conjugate diameters, of lengths $2a_1$, $2a_2$, $2a_3$,

as coordinate axes. And let $(\Xi, \mathrm{H}, \mathrm{Z})$ be the coordinates of the corresponding point on the sphere, referred to the corresponding diameters of the sphere as axes. Now it is a property of our transformation that parallel lines correspond to parallel lines with the ratio of their lengths unchanged. Therefore

$$\xi/a_1 = \Xi, \quad \eta/a_2 = \mathrm{H}, \quad \zeta/a_3 = \mathrm{Z}.$$

Also $$\Xi^2 + \mathrm{H}^2 + \mathrm{Z}^2 = 1,$$

therefore $$\xi^2/a_1{}^2 + \eta^2/a_2{}^2 + \zeta^2/a_3{}^2 = 1.$$

This is the equation of the ellipsoid referred to any set of conjugate diameters as coordinate axes. The form of it might be obtained as a direct deduction from the symmetry of the surface with reference to conjugate diameters.

77. *Circular Sections.* Draw a sphere

$$x^2 + y^2 + z^2 = r^2,$$

intersecting the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

for which $a > b > c$. Thus r must not be greater than a or less than c. If $r = a$ the curve of intersection is reduced to two single points. If r is between a and b it consists of two loops encircling the axis of x. If r is between b and c it consists of two loops encircling the axis of z. If $r = c$ it is reduced again to two single points. The loops may be drawn so as to cover the ellipsoid, each pair of loops corresponding to a certain length of radius of the sphere; and at every point of each loop the curve is at right angles to the radius drawn from the centre to the point in question, because it is a curve on the sphere. The only case not dealt with is that in which $r = b$. It will now be proved that in this case the sphere intersects the ellipsoid in two circles intersecting on the axis of y.

If r is not greater than a, or less than c, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=\frac{1}{r^2}(x^2+y^2+z^2-r^2),$$

or $$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z^2=0$$

represents a cone, with vertex at the origin, drawn through the points of intersection of the ellipsoid and the sphere; because it is the equation of a cone, (§ 24), and is satisfied at all points common to these two surfaces. Now there is one, and only one, value of r, namely b, which makes this cone a real pair of planes, (§ 45). The equation of this pair of planes is

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)x^2-\left(\frac{1}{c^2}-\frac{1}{b^2}\right)z^2=0;$$

and these planes, passing through the axis of y, meet the sphere, and therefore the ellipsoid, in two circles, the centres of which are at the origin. Accordingly they give the only central circular sections of the ellipsoid. Therefore the only plane circular sections of the ellipsoid are the sections by these planes and planes parallel to them. They form two systems of circular sections, which may be distinguished by calling them opposite systems.

The cases of $r=a$ and $r=c$ might be regarded as giving imaginary plane sections, but here we are concerned only with real points. This is one of the cases in which we meet with the lack of symmetry which is the natural result of confining our attention to real figures.

78. *Axes of a Plane Section.* The same construction gives the axes of any central plane section, as follows. Take any point P, (x', y', z'), on the ellipsoid; and let r be the length of the central radius OP; and draw the cone

$$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z^2=0,$$

cutting the ellipsoid in the loop through the point P; and draw the tangent plane of the cone at P. Then OP is one

of the axes of the section of the ellipsoid by this plane, because it is at right angles to the tangent of this section at P. But the equation, (§ 25), of the tangent plane of the cone at the point P, is

$$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x'x+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y'y+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z'z=0.$$

Therefore in order to find the axes of the section of the ellipsoid by a given plane through the centre, namely

$$\lambda x+\mu y+\nu z=0,$$

we have to find r such that

$$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x',\quad \left(\frac{1}{b^2}-\frac{1}{r^2}\right)y',\quad \left(\frac{1}{c^2}-\frac{1}{r^2}\right)z'$$

are proportional to λ, μ, ν. Now x', y', z' satisfy the equation

$$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x'^2+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y'^2+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z'^2=0,$$

therefore $$\frac{a^2\lambda^2}{r^2-a^2}+\frac{b^2\mu^2}{r^2-b^2}+\frac{c^2\nu^2}{r^2-c^2}=0.$$

This is a quadratic equation for r^2, whose roots, r_1^2 and r_2^2, are the squares of the two semi-axes of the given section. If λ, μ, ν are actual direction cosines, it gives

$$r_1^2r_2^2=\frac{a^2b^2c^2}{a^2\lambda^2+b^2\mu^2+c^2\nu^2},$$

which agrees with the formula given in § 75.

This investigation shows that the direction cosines of a semi-axis, of length r_1, of the section of the ellipsoid by the plane

$$\lambda x+\mu y+\nu z=0,$$

are proportional to x', y', z', when

$$\left(\frac{1}{a^2}-\frac{1}{r_1^2}\right)x',\quad \left(\frac{1}{b^2}-\frac{1}{r_1^2}\right)y',\quad \left(\frac{1}{c^2}-\frac{1}{r_1^2}\right)z'$$

are proportional to λ, μ, ν. Therefore the direction cosines of this semi-axis are proportional to

$$\frac{a^2\lambda}{r_1^2-a^2},\quad \frac{b^2\mu}{r_1^2-b^2},\quad \frac{c^2\nu}{r_1^2-c^2}.$$

Having found the semi-axes, r_1, r_2, of a central plane section of the ellipsoid, we can now find the semi-axes, r_1', r_2', of the section by any parallel plane; p' being the perpendicular from the centre on this plane, and p the perpendicular from the centre on the parallel tangent plane. To do this, consider the corresponding sections of the corresponding sphere, of radius unity, in the second figure. One of them is a central section of radius 1; and the other is a section by plane at distance d from the centre, of radius $\sqrt{(1-d^2)}$. The relations between the two figures are

$$\frac{d}{1}=\frac{p'}{p}, \text{ and } \frac{\sqrt{(1-d^2)}}{1}=\frac{r_1'}{r_1}=\frac{r_2'}{r_2};$$

therefore

$$r_1' = r_1\sqrt{\left(1-\frac{p'^2}{p^2}\right)} \text{ and } r_2' = r_2\sqrt{\left(1-\frac{p'^2}{p^2}\right)}.$$

79. *Poles and Polars.* The pole and polar properties of a sphere, investigated in a previous chapter, can be extended to an ellipsoid by omitting all references to right angles. A point $(x'/a, y'/b, z'/c)$ in the second figure, and its polar plane

$$\frac{x'}{a}X+\frac{y'}{b}Y+\frac{z'}{c}Z=1,$$

with reference to the sphere $X^2+Y^2+Z^2=1$, correspond to a point (x', y', z'), and the plane

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1,$$

in the first figure. This plane is called the polar plane of the point (x', y', z') with reference to the ellipsoid, and the point is called the pole of the plane. And, as before, if a point Q is on the polar plane of a point P, P is on the polar plane of Q. The equation

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{2}{\rho_3},$$

(§ 62), applies to the first figure without alteration, being a relation between the ratios of segments of a straight line.

This shows, as before, that the relation of a point to its polar plane is independent of the choice of coordinate axes; and gives the proposition that the polar plane of an external point is the plane of contact of an enveloping cone with vertex at that point.

We still have pairs of polar lines, corresponding to polar lines with regard to the sphere, and defined in the same way, (§§ 65, 137). But, in general, two polar lines are not at right angles, because the normals at the points in which one of them cuts the ellipsoid do not in general intersect.

80. *Director Sphere.* Let p_1, p_2, p_3 be the lengths, and $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ the direction cosines, of the perpendiculars from the centre on three tangent planes of the ellipsoid at right angles to one another. Then

$$p_1{}^2 = a^2\lambda_1{}^2 + b^2\mu_1{}^2 + c^2\nu_1{}^2,$$
$$p_2{}^2 = a^2\lambda_2{}^2 + b^2\mu_2{}^2 + c^2\nu_2{}^2,$$
$$p_3{}^2 = a^2\lambda_3{}^2 + b^2\mu_3{}^2 + c^2\nu_3{}^2.$$

Now $(\lambda_1, \lambda_2, \lambda_3)$, (μ_1, μ_2, μ_3), (ν_1, ν_2, ν_3) are sets of direction cosines, for they would be the direction cosines of the principal axes if the three perpendiculars were taken as coordinate axes. Therefore we get by addition

$$p_1{}^2 + p_2{}^2 + p_3{}^2 = a^2 + b^2 + c^2.$$

Also if Q is the point of intersection of the tangent planes,

$$OQ^2 = p_1{}^2 + p_2{}^2 + p_3{}^2;$$

therefore $$OQ^2 = a^2 + b^2 + c^2.$$

Therefore the locus of the intersection of three tangent planes at right angles to one another is a sphere, the square of whose radius is $a^2 + b^2 + c^2$. It is called the director sphere of the ellipsoid.

81. *Spheroids.* If two of the principal axes of an ellipsoid are equal, the surface is called an ellipsoid of revolution, or a spheroid. Its equation may be written

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

It is called a prolate spheroid if $c > a$, and an oblate spheroid if $c < a$. It may be generated by the revolution of an ellipse about one of its principal axes. The circular sections are the sections by planes at right angles to this axis. It is obvious that some of the properties of a general ellipsoid lose their meaning in consequence of this simplification; and that some properties of the surface are immediately derivable from conics.

82. *Examples.*

1. Find the locus of the centres of the sections of an ellipsoid by planes through a given point. [Take the case of a sphere, centre O, and a given point P; then it is obvious that the centres of the sections are on a sphere, diameter OP. Therefore for the ellipsoid, the centres are on a similar and similarly situated ellipsoid; of which the line joining the centre of the given ellipsoid to the given point is a diameter.] (S 1.)

2. Any two circular sections of an ellipsoid, of opposite systems, are on a sphere. [The following procedure is applicable to any quadric. The general equation of a pair of planes giving these sections is

$$\left\{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^{\frac{1}{2}} x+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^{\frac{1}{2}} z-A\right\}\left\{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^{\frac{1}{2}} x-\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^{\frac{1}{2}} z-B\right\}=0,$$

or $\dfrac{1}{b^2}(x^2+y^2+z^2)-\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)$

$$-(A+B)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^{\frac{1}{2}} x+(A-B)\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^{\frac{1}{2}} z+AB-1=0,$$

and the form of this equation shows that all points of intersection of either of these planes by the ellipsoid are on a certain sphere.]

3. Two circular sections of opposite systems are such that the sum of the squares of their distances from the centre of the ellipsoid is constant; prove that the centre of the sphere on which they both lie is on a fixed ellipse. (S.)

4. The enveloping cone is drawn to the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

from the point (x, y, z). Prove that the volume between the cone and the nearer part of the surface of the ellipsoid is

$$\tfrac{1}{3}\pi abc\,\{(x^2/a^2+y^2/b^2+z^2/c^2)^{\frac{1}{2}}+(x^2/a^2+y^2/b^2+z^2/c^2)^{-\frac{1}{2}}-2\}. \quad \text{(C.)}$$

5. For the same ellipsoid, show that the radius of each circular section by a plane through the point $(a, 0, 0)$ is $(b^2-c^2)^{\frac{1}{2}}(a^2-c^2)^{-\frac{1}{2}}b$. (S 1.)

6. For the same ellipsoid, tangent planes and normals are drawn at all points for which $z = h$. Show that the lines in which the tangent planes cut the plane $z = 0$ are tangents of the ellipse

$$x^2/a^2 + y^2/b^2 = c^2/(c^2 - h^2).$$

And show that the normals meet the plane $z = 0$ in an ellipse which is similar to $x^2/a^2 + y^2/b^2 = 1$ if $a = b$, or if $a^2 + b^2 = 2c^2$, or if $1/a^2 + 1/b^2 = 2/c^2$. (S 1.)

7. For the same ellipsoid, prove that one of the angles between the two tangent planes which pass through the line $x \sin \alpha = y \cos \alpha$, $z = k$, is 2ϕ, where

$$\tan^2 \phi = (k^2 - c^2)(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)^{-1}. \qquad \text{(S 1.)}$$

8. For the same ellipsoid, prove that the plane section whose centre is at the point $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$ passes through three of the extremities of the principal axes. (S 1.)

9. Prove that if the chord joining two points on the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

touches the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = \tfrac{1}{2},$$

the two points must lie at the extremities of conjugate diameters, and the point of contact must bisect the chord. (S 1.)

10. Prove that the straight lines through a given point, (x_1, y_1, z_1), which are at right angles to their polar lines with respect to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, are generators of the cone

$$a^2(x - x_1)(y_1 z - z_1 y) + b^2(y - y_1)(z_1 x - x_1 z) + c^2(z - z_1)(x_1 y - y_1 x) = 0.$$

(S 1.)

11. A line in the fixed plane $z = h$ is such that two perpendicular planes can be drawn through it to touch the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Prove that the line always touches an ellipse whose semi-axes are

$$(a^2 + c^2 - h^2)^{\frac{1}{2}}, \quad (b^2 + c^2 - h^2)^{\frac{1}{2}}. \qquad \text{(S 1.)}$$

12. The sum of the squares of the perpendiculars from a point to the lines $y = x \tan \theta$, $z = c$ and $y = -x \tan \theta$, $z = -c$ being given, prove that the locus of the point is an ellipsoid, and find the lengths of its principal axes. (S 1.)

13. A, B, C are the extremities of conjugate radii of an ellipsoid with centre O; show that the area of the projection of the triangle OBC on any principal plane is proportional to the projection of OA on the corresponding axis. Show also that if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the

coordinates of A, B, C, referred to the principal axes, the coordinates of the pole, P, of the plane ABC are

$$x_1 + x_2 + x_3, \quad y_1 + y_2 + y_3, \quad z_1 + z_2 + z_3;$$

and that the equations of the polar line of AP are

$$\frac{x - x_1}{x_2 - x_3} = \frac{y - y_1}{y_2 - y_3} = \frac{z - z_1}{z_2 - z_3}. \qquad \text{(S 1.)}$$

14. Normals of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, at the points in which it cuts the plane $lx + my + nz = p$, are drawn. Show that the points in which they meet the plane $x = 0$ lie on a conic. (S 1.)

15. Normals are drawn to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ making a constant angle, α, with the plane of xy. Prove that they cut this plane on the ellipse

$$\frac{a^2 + c^2 \tan^2 \alpha}{(a^2 - c^2)^2} x^2 + \frac{b^2 + c^2 \tan^2 \alpha}{(b^2 - c^2)^2} y^2 = 1. \qquad \text{(S 1.)}$$

16. Find the equation of the enveloping cylinder of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the direction cosines of whose axis are (l, m, n). And prove that the area of the section of this cylinder by a plane perpendicular to its axis is

$$\pi abc\,(l^2/a^2 + m^2/b^2 + n^2/c^2)^{\frac{1}{2}}. \qquad \text{(C.)}$$

17. A straight rod moves so that three given points on it are in three given planes, which meet in a point. Find the equation of the surface traced out by another given point on the rod.

18. Find the lengths of the axes of the ellipse which is the section of the surface $x^2 + 2y^2 + 3z^2 = 1$ by the plane $x + y + z = 0$.

19. Prove that, if

$$\frac{ll'}{a^2(b^2 - c^2)} = \frac{mm'}{b^2(c^2 - a^2)} = \frac{nn'}{c^2(a^2 - b^2)},$$

the lines $x/l = y/m = z/n$ and $x/l' = y/m' = z/n'$ are principal axes of a section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. (S 1.)

CHAPTER VII

QUADRIC SURFACES OF REVOLUTION

83. Two classes of quadric surfaces, namely cylinders and surfaces of revolution, are immediately derivable from conics in such a way as to secure that the enumeration of them is complete.

The case of cylinders is obvious. For an equation of the second order, in x and y coordinates, defines a conic in plane geometry; and the same equation in solid geometry represents any form of quadric cylinder.

The derivation from conics of all the real quadric surfaces of revolution will now be undertaken. And this having been accomplished, quadric surfaces of a more general character will be found by means of a transformation, (§ 56), of the form

$$x = aX, \quad y = bY, \quad z = cZ.$$

It will be proved in §§ 125–127 that the enumeration of real quadric surfaces which is obtained by following this procedure, showing their forms very clearly, is complete.

84. *Surfaces of Revolution.* A surface which can be generated by the revolution of a plane curve, about a straight line in its plane, is called a surface of revolution. The straight line is called the axis of the surface, or its axis of revolution; and the curve is called the generating curve. A surface generated in this way is symmetrical with regard to the axis. Thus it can also be generated, in a variety of ways, by the revolution, about the axis, of a curve which is not in a plane through the axis, and need not be a plane curve.

The term surface of revolution may also be applied to the locus of any equation which is symmetrical with reference to a straight line, without regard to whether it is

real or imaginary. But this does not concern us so long as we are dealing only with the forms of real figures.

It is stipulated that the axis is not to be at infinity. Thus a surface which is the limiting case of a surface of revolution, when the axis passes to infinity, is not on that account classed as a surface of revolution.

85. To find all the quadric surfaces of revolution, take the axis of the surface for the axis of z. Then the equation of a surface of revolution must involve x and y only in the form $x^2 + y^2$, and must therefore be

$$\lambda (x^2 + y^2) = Az^2 + Bz + C,$$

if it is to be an equation of the second order. And the equation of the generating curve in the plane of zx is

$$\lambda x^2 = Az^2 + Bz + C.$$

The real surfaces represented by this equation, (other than a single line or single point), will now be classified.

(i) If $\lambda = 0$ we get a real surface, namely a pair of parallel or coincident planes, if $B^2 - 4AC$ is not negative. In all other cases λ can be, and will be, taken to be unity.

(ii) If A and B are both zero C must be positive, and the equation may be written

$$x^2 + y^2 = a^2,$$

representing a circular cylinder of radius a.

(iii) If A is zero, and B is not zero, the generating curve is the parabola $x^2 = Bz + C$; and the surface, whose equation is

$$x^2 + y^2 = Bz + C,$$

is generated by the revolution of a parabola about its axis. It is called a paraboloid of revolution. By shifting the origin to the point $(0, 0, -C/B)$, the equation is put into one of the two forms

$$x^2 + y^2 = \pm 2lz,$$

where $2l$ is the length of the latus rectum of the parabola. The new origin is called the vertex of the surface.

(iv) We have now dealt with all the cases in which A is zero. If A is not zero, the equation of the surface may be written

$$x^2 + y^2 = A\left(z + \frac{B}{2A}\right)^2 - \frac{B^2}{4A} + C.$$

Let us shift the origin to the point $(0, 0, -B/2A)$; the equation then takes the form

$$x^2 + y^2 = Az^2 + D,$$

and the equation of the generating curve is $x^2 - Az^2 = D$. If D is zero, A must be positive, and the generating curve is a pair of straight lines equally inclined to the axis of z; thus the surface is a cone of revolution whose equation may be written $$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 0.$$

(v) Finally, if A and D are neither of them zero, the generating curve, $x^2 - Az^2 = D$, is either an ellipse (which may be a circle), or a hyperbola; one of the principal axes being the axis of revolution. And the equation of the surface assumes one of the three possible forms of

$$\frac{x^2+y^2}{a^2} \pm \frac{z^2}{c^2} = \pm 1.$$

The surface $$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$$

is an ellipsoid of revolution. It is a prolate spheroid if $a^2 < c^2$, an oblate spheroid if $a^2 > c^2$, (§ 81), and a sphere if $a^2 = c^2$.

The surface $$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1$$

is formed by the revolution of a hyperbola about its conjugate axis, and is called a hyperboloid of revolution of one sheet.

The surface $$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = -1$$

is formed by the revolution of a hyperbola about its transverse axis, and is called a hyperboloid of revolution of two sheets.

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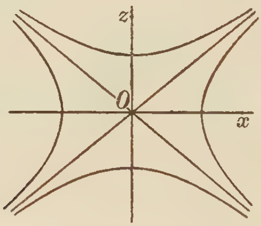
The real quadric surfaces of revolution have now been enumerated completely.

In each case in which A is not zero the new origin is called the centre of the surface; and is a unique point which, except when it is the vertex of a cone, bisects all chords drawn through it. In the case of a circular cylinder every point on the axis bisects all chords drawn through it, and may be regarded as a centre of the surface. In the case of a pair of parallel planes every point on the plane halfway between them has the same property. The properties of a parabola show that, for a paraboloid of revolution, there is no point which bisects all chords drawn through it; therefore this surface is said to have no centre.

86. *The Hyperboloids of Revolution.* The two hyper-



boloids can be generated together. Draw in the plane of zx the hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1,$$

and its conjugate

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1,$$

and their asymptotes

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0.$$

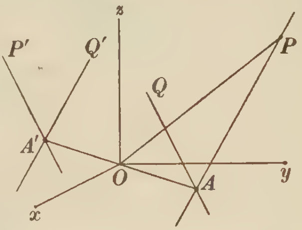
Revolution of this figure, about the axis of z, generates the hyperboloids of one and two sheets whose equations are written above, and the cone

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=0,$$

which is called the asymptotic cone of each of the two hyperboloids.

These two hyperboloids are said to be conjugate to one another, and the cone is an asymptotic cone common to them. The shapes of the hyperboloids are easily perceived. The first is a single continuous surface, and the second is divided into two separate portions.

87. *Rectilinear Generators.* It will now be proved that the hyperboloid of revolution of one sheet is the surface generated by the revolution, about an axis, of a straight line which does not meet it, and is not parallel to it.



Take the axis of revolution for axis of z, and the origin at one extremity of the shortest distance, OA, between the axis of z and the revolving line, AP. Let $OA = a$, and let α be the constant acute angle of inclination of AP to Oz. Thus OA revolves in the plane of xy, and if P, (x, y, z), is any point on the surface generated,

$$AP = z \sec \alpha, \quad \text{and} \quad OP^2 = OA^2 + AP^2.$$

Therefore $$x^2 + y^2 + z^2 = a^2 + z^2 \sec^2 \alpha,$$

or $$x^2 + y^2 - z^2 \tan^2 \alpha = a^2.$$

This is the equation of the surface generated, which is therefore a hyperboloid of revolution of one sheet. Its equation may be written

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1,$$

where $c = a \cot \alpha$.

All the straight lines, AP, lie on the surface; also another system of straight lines, AQ, at right angles to OA, and inclined at the same angle, α, to the axis, but sloping the other way. This is obvious from the symmetry of the figure with regard to the axis. By means of this second system of lines the surface might equally well have been

generated. These two systems of lines are called the generating lines, or rectilinear generators, of the surface. No other straight line can lie on the surface, because if it did the surface could be generated by the revolution of this line. But generation by this line would give different values for $\tan^2 \alpha$ and a^2, or would require a different position for the origin, and would therefore give a different surface.

The two systems of rectilinear generators, of which AP and AQ are samples, cover the surface with what may be called a network of lines, one line of each system passing through any given point of it.

The symmetry of the arrangement shows that each line of one system meets every line of the other system, except one which is parallel to it. The line of the second system which is parallel to AP in the diagram is the line $A'Q'$ drawn through the point A', in AO produced.

The arrangement of the lines can be seen clearly by drawing a diagram showing the orthogonal projections of all the lines on the plane of xy. These projections are the tangents of a circle of radius a, each tangent representing two generators, one of each system.

The two systems of lines on a given surface are distinguished by calling them opposite systems.

88. To find the equations of the rectilinear generators of the surface

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1,$$

drawn through any given point, A, of the section of the surface by the plane of xy, let ϕ be the angle xOA, and (l, m, n) the direction cosines of a generator through A. Then the coordinates of A are $(a \cos \phi,\ a \sin \phi,\ 0)$; and $n = \cos \alpha$, where α is the acute angle given by $c = a \cot \alpha$. And as each generator is at right angles to OA,

$$l \cos \phi + m \sin \phi = 0;$$

therefore

$$l = \pm \sin \alpha \sin \phi, \quad m = \mp \sin \alpha \cos \phi, \quad n = \cos \alpha.$$

The upper signs give the direction cosines of one of the two generators through A, and the lower signs give the direction cosines of the other. Therefore the equations of a generator are

$$\frac{x - a\cos\phi}{\sin\alpha\sin\phi} = \frac{y - a\sin\phi}{-\sin\alpha\cos\phi} = \pm\frac{z}{\cos\alpha},$$

or

$$\frac{x - a\cos\phi}{a\sin\phi} = \frac{y - a\sin\phi}{-a\cos\phi} = \pm\frac{z}{c}.$$

89. *Transformation by Homogeneous Strain.* From each of the quadric surfaces of revolution, except the pair of parallel planes, a quadric surface of a more general character can be derived by a homogeneous strain defined by the equations of transformation, (§ 55),

$$x = X, \quad y = \frac{a}{b}Y, \quad z = Z.$$

This transformation has the effect of converting each expression $(x^2 + y^2)/a^2$, in the equations of the surfaces of revolution, into $x^2/a^2 + y^2/b^2$ (the special notation, X, Y, Z, being dropped after the substitution). Thus the circular sections, by planes at right angles to the axis of z, are converted into a set of similar ellipses.

The cylinder, (§ 85, ii), becomes the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The paraboloid of revolution, (§ 85, iii), becomes a surface which is called an elliptic paraboloid, whose equation has one of the two forms

$$\frac{x^2}{l} + \frac{y^2}{l'} = \pm 2z,$$

where the ratio $l : l'$ is equal to the ratio $a^2 : b^2$, and $2l$, $2l'$ are the lengths of the latera recta of the parabolic sections by the planes of zx and yz. The axis of z is called the axis of this surface.

The cone of revolution, (§ 85, iv), becomes the cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The ellipsoid of revolution, (§ 85, v), becomes the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The two hyperboloids of revolution (§ 85, v) give the more general surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,$$

which are called respectively the hyperboloid of one sheet, and the hyperboloid of two sheets.

Also the hyperboloid of one sheet has rectilinear generators, because straight lines in the old figure correspond to straight lines in the new one (§ 56). And the equations of these lines, drawn through a point ($a \cos \phi$, $b \sin \phi$, 0), are

$$\frac{x - a\cos\phi}{a\sin\phi}=\frac{y - b\sin\phi}{-b\cos\phi}=\pm\frac{z}{c};$$

the upper sign giving all the generators of one system, and the lower sign giving all the generators of the other system.

The general character of each surface, apart from the question of symmetry about an axis, is unaffected by the strain. And as straight lines in the new figure correspond to straight lines in the old one, no surface acquires the property of having straight lines on it which did not possess the property before.

90. *Conjugate Hyperboloids.* The two hyperboloids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1,$$

one of them a hyperboloid of one sheet, and the other a hyperboloid of two sheets, are said to be conjugate to one another, and have a common asymptotic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

The shapes of these surfaces are easily derived from

those of the corresponding conjugate hyperboloids of revolution, and their asymptotic cone; the introduction of a set of similar ellipses, in the place of the circular sections, being a change which does not cause any difficulty.

91. *The Hyperbolic Paraboloid.* From any one of these quadric surfaces we may now expect to be able to derive another one, which may be sufficiently distinct to deserve a distinct name, by taking the limiting case in which elliptic or hyperbolic sections by a series of parallel planes become parabolas. The ellipsoid and the hyperboloid of two sheets both give in this way the elliptic paraboloid, which has been derived independently from the parabola. But the hyperboloid of one sheet gives a new surface, which is called a hyperbolic paraboloid. It is a particular limiting case of a hyperboloid of one sheet, and therefore retains the property of that surface of having straight lines on it.

To derive the equation of the hyperbolic paraboloid from that of the hyperboloid of one sheet,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

we have to take the limiting case in which the elliptic section by the plane of xy becomes a parabola with latus rectum $2l$, and at the same time the hyperbolic section by the plane of zx becomes a parabola with latus rectum $2l'$. That is to say we must take the limiting case in which a, b and c all tend to infinity, in such a manner that b^2/a has a finite limit l, and c^2/a has a finite limit l'. In order to investigate the result of this, it is necessary to shift the origin to the point $(-a, 0, 0)$. Referred to the new axes thus taken, the equation of the hyperboloid is

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

or $$\frac{a}{b^2}y^2-\frac{a}{c^2}z^2=2x-\frac{x^2}{a}.$$

We can now proceed to the limit, with the result that the equation becomes

$$\frac{y^2}{l} - \frac{z^2}{l'} = 2x,$$

where l and l' are positive. The line which is here taken for axis of x is called the axis of the surface; and the origin is called the vertex.

92. The shapes of the hyperboloids are easily derived from a conception of the revolution of a hyperbola about its conjugate or its transverse axis. But it is not easy to perceive the shape of the hyperbolic paraboloid, or to draw a diagram of it. Accordingly the following method of constructing this surface may be helpful.

Imagine an infinite number of cards, all alike, in the shape of a parabola, extending to infinity, the part cut away being that which is on the convex side of the parabola. Let the cards be placed in a pile, like the leaves of a book, with all the axes of the parabolas in a plane at right angles to the cards. If the vertices of the cards are in a straight line, the surface of the pile is a parabolic cylinder. But if the vertices of the cards are on a parabola with its axis in one of the cards, the pile forms a model of a paraboloid. If the cards are on the concave side of this parabola, the paraboloid is elliptic, and has only elliptic and parabolic sections. If the cards are on the convex side the paraboloid is hyperbolic, and has only hyperbolic and parabolic sections. If the two parabolas employed in this construction are equal, that is to say if $l = l'$, the surface has the property that the two sides of it are identical. If a model of this surface were made, in the form of a thin sheet, the two sides of it would not be distinguishable by shape.

93. *Synclastic and Anticlastic Curvature.* The curvature of a hyperboloid of one sheet, or of a hyperbolic paraboloid, at every point of it, is said to be anticlastic, to distinguish it from the curvature of an ellipsoid, or a hyperboloid of two sheets, which is said to be synclastic.

The shape of a saddle is a familiar example of anticlastic curvature. At the middle of a saddle, the surface of it curves upwards in front and behind, and downwards towards each side. Accordingly a surface with anticlastic curvature is sometimes referred to as saddle shaped. An anticlastic surface may also be illustrated by the top of a mountain pass, where the path goes down in front and behind, and hills rise on either hand. On a map contour lines intersect there.

94. *Notes and Examples.*

1. The coordinates of the point of intersection of two given generators of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, one of each system, and one of them specified by ϕ and the other by ϕ', (§ 89), are found by solving the equations

$$(x - a\cos\phi)/a\sin\phi = -(y - b\sin\phi)/b\cos\phi = z/c,$$

$$(x - a\cos\phi')/a\sin\phi' = -(y - b\sin\phi')/b\cos\phi' = -z/c$$

for x, y and z. Dealing first with x and y we have

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1, \qquad \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1,$$

which gives

$$x = a\frac{\cos\frac{1}{2}(\phi' + \phi)}{\cos\frac{1}{2}(\phi' - \phi)}, \qquad y = b\frac{\sin\frac{1}{2}(\phi' + \phi)}{\cos\frac{1}{2}(\phi' - \phi)};$$

and the equation of the surface gives

$$z^2/c^2 = x^2/a^2 + y^2/b^2 - 1 = \sec^2\tfrac{1}{2}(\phi' - \phi) - 1 = \tan^2\tfrac{1}{2}(\phi' - \phi);$$

therefore $z = \pm c\tan\frac{1}{2}(\phi' - \phi)$.

Let us find the points, if any, at which the two generators are at right angles. The condition for this is

$$a^2\sin\phi\sin\phi' + b^2\cos\phi\cos\phi' - c^2 = 0,$$

which gives

$$a^2\cos^2\tfrac{1}{2}(\phi - \phi')(1 - x^2/a^2) + b^2\cos^2\tfrac{1}{2}(\phi - \phi')(1 - y^2/b^2) - c^2 = 0.$$

Therefore $$a^2 - x^2 + b^2 - y^2 - c^2(z^2/c^2 + 1) = 0$$

or $$x^2 + y^2 + z^2 = a^2 + b^2 - c^2.$$

Thus the points in question, if they exist, are the points of intersection of the surface by the sphere represented by this equation.

2. For the hyperbolic paraboloid $y^2/l - z^2/l' = 2x$, the general equation of a generating line of one system is

$$\frac{x}{2\lambda} = \frac{y - \lambda\sqrt{l}}{\sqrt{l}} = \frac{z + \lambda\sqrt{l'}}{\sqrt{l'}},$$

and the general equation of a generating line of the other system is

$$\frac{x}{2\mu} = \frac{y - \mu\sqrt{l}}{\sqrt{l}} = -\frac{z - \mu\sqrt{l'}}{\sqrt{l'}}.$$

This can be proved by substituting for y and z in terms of x in the equation of the surface, when it will be seen that every point of each line is on the surface for all values of λ and μ. And by solving the equations of these lines for x, y and z, their point of intersection is found to be

$$x = 2\lambda\mu, \quad y = \sqrt{l}\,(\lambda + \mu), \quad z = \sqrt{l'}\,(\mu - \lambda).$$

The two systems of lines may be called the λ system and the μ system. Every line of the λ system is parallel to the plane $\sqrt{l'}y - \sqrt{l}\,z = 0$, and every line of the μ system is parallel to the plane $\sqrt{l'}\,y + \sqrt{l}\,z = 0$.

3. Prove that the points on the surface $y^2/l - z^2/l' = 2x$ at which the generating lines are at right angles are in a plane parallel to the plane of yz, and at a distance from it equal to $\pm \frac{1}{2}(l' - l)$.

4. Prove that the generating lines through a point P on the section $z = c$ of the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ meet the principal section $z = 0$ at the ends of a pair of conjugate diameters. (S.)

5. Prove that the tangent of the angle between the generating lines of the paraboloid $x^2 - y^2 = 2z$, at the point (x, y, z), is $(x^2 + y^2 + 1)/z$.

6. Find the equation of the surface generated by a straight line which passes through the two lines $\lambda x = y$, $z = 0$ and $\mu x = y$, $z = k$, and is parallel to a plane whose direction cosines are (l, m, n). (C.)

7. A plane through the vertex cuts the paraboloid

$$x^2/a^2 + y^2/b^2 + 2z/c = 0$$

so that the area of the projection of the section on the plane $z = 0$ is constant, and equal to πab. Show that the plane envelops the cone

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0. \qquad \text{(C.)}$$

8. Show that the projections on the plane of xy of the generating lines of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ are the tangents of the section by this plane.

9. Prove that the angle between the generators of the surface $x^2/a - y^2/b = 2z$ at the point (x, y, z) is

$$\tan^{-1}\frac{\left\{ab\left(1+\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right\}^{\frac{1}{2}}}{z+\frac{1}{2}(a-b)}. \qquad \text{(C.)}$$

10. Show that the generators of $x^2/a^2 + y^2/b^2 = 2z/c$, through the point (x_0, y_0, z_0) on the surface, lie at the intersections of the tangent plane with the planes

$$(xy_0 - yx_0)^2/a^2b^2 = (z - z_0)^2/c^2. \qquad \text{(C.)}$$

11. Show that the six coordinates, $(l, m, n, \lambda, \mu, \nu)$, of the generating lines of the paraboloid $x^2/a^2 - y^2/b^2 = 2z/c$ satisfy the equations

$$\lambda a \pm \mu b = 0, \quad lb \pm ma = 0, \quad c\nu \pm abn = 0. \qquad \text{(C.)}$$

12. The well known construction for the foci and directrices of a conic, which is a plane section of a cone of revolution, applies also to any quadric of revolution within which a sphere can be inscribed.

To prove this, take $x^2 + y^2 + z^2 = a^2$ as the equation of a sphere. Then the general equation of a quadric of revolution in which it is inscribed is

$$x^2 + y^2 + z^2 - a^2 = k(z-b)^2,$$

this being the limiting case, when $b = c$, of the equation

$$x^2 + y^2 + z^2 - a^2 = k(z-b)(z-c).$$

And as the quadric is outside the sphere, k is positive. Let us write the equation $T^2 = kL^2$. Then T is the length of a tangent, drawn to the sphere, from a point, (x, y, z), on the surface; and L is the perpendicular distance of this point from the plane of contact, $z - b = 0$.

Consider now a section of the surface by a plane touching the sphere at a point S, and inclined to the plane $z = b$ at an angle θ, and cutting this plane in a line AB. The equation of the surface states that, if P is any point on the section, and PM the perpendicular on the line AB, PS is equal to $\sqrt{k}\sin\theta \times PM$. That is to say the section is a conic, of which S is the focus, and AB the corresponding directrix, and $\sqrt{k}\sin\theta$ the eccentricity.

CHAPTER VIII

QUADRIC CONE

95. The general homogeneous equation of the second order will be written

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

or
$$\phi(x, y, z) = 0.$$

This equation has a real locus, for it is satisfied at the origin. If it is satisfied at any other point, (x', y', z'), it is also satisfied at every point whose coordinates are $(\lambda x', \lambda y', \lambda z')$, that is to say every point on the straight line through (x', y', z') and the origin. Therefore the locus is either a single point, namely the origin, for it is obvious that this is possible, or else a cone with a vertex at the origin, see § 24. And for the same reason, the equation of a cone, referred to a vertex as origin, must be homogeneous; and therefore of the form $\phi(x, y, z) = 0$ if it is a quadric surface.

The equation in terms of polar coordinates, r, l, m, n, is

$$r^2 \phi(l, m, n) = 0.$$

At the origin r is zero. If the surface is a cone, the direction cosines of the generating lines satisfy the equation $\phi(l, m, n) = 0$, which often serves as a convenient specification of the surface.

96. *Cone which is also a Cylinder.* Let us find the condition that the surface $\phi(x, y, z) = 0$ is a cone with more than one vertex. Draw through the origin a straight line with direction cosines (l, m, n), and shift the origin to a point $(l\rho, m\rho, n\rho)$ on this line. The new equation of the surface is

$$\phi(x + l\rho,\ y + m\rho,\ z + n\rho) = 0,$$

or
$$\phi(x, y, z) + 2\rho(Ax + By + Cz) + \rho^2 \phi(l, m, n) = 0;$$

where
$$A = al + hm + gn,$$
$$B = hl + bm + fn,$$
$$C = gl + fm + cn,$$
and
$$\phi(l, m, n) = Al + Bm + Cn.$$

The surface is a cone of which the new origin, as well as the old one, is a vertex if, and not unless, the equation so treated retains its homogeneous form; that is to say if, and not unless, A, B and C are all zero. This condition is independent of the value of ρ; therefore, if it is satisfied, a straight line exists every point of which is a vertex, and the equation of the surface is not changed by shifting the origin along this line. Thus the surface must be a cylinder; in fact it must be either a single straight line, or else a pair of planes either intersecting or coincident, see § 24.

Accordingly the direction cosines of a line of vertices are given by the equations $A = 0$, $B = 0$, $C = 0$, or
$$al + hm + gn = 0,$$
$$hl + bm + fn = 0,$$
$$gl + fm + cn = 0.$$

Therefore such a line exists if, and not unless, these equations can be satisfied simultaneously; that is to say if, and not unless,
$$abc + 2fgh - af^2 - bg^2 - ch^2$$
is zero. This expression, which may be written
$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$
is called the discriminant of $\phi(x, y, z)$, and will be denoted by Δ.

When this condition is satisfied, the equations $A = 0$, $B = 0$, $C = 0$ give, in general, a single direction for a

line of vertices. But exceptionally, as in the case in which the three equations coincide, they give any direction in a certain plane.

97. *Change of Directions of the Axes.* Let us now write down the result of the general transformation for a change of the directions of the axes, (§ 50). By this transformation, $\phi(x, y, z)$ becomes

$$\phi(l_1X + l_2Y + l_3Z,\ m_1X + m_2Y + m_3Z,\ n_1X + n_2Y + n_3Z),$$

or $$a'X^2 + b'Y^2 + c'Z^2 + 2f'YZ + 2g'ZX + 2h'XY;$$

which will be written $\psi(X, Y, Z)$, X, Y, Z being the coordinates of a point referred to the new axes. Also the first three of the new coefficients are easily written down, as follows:

$$a' = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1,$$

$$b' = al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2,$$

$$c' = al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3l_3 + 2hl_3m_3.$$

Therefore, taking account of the relations between the direction cosines of three lines at right angles to one another, (§ 50), we get by addition

$$a' + b' + c' = a + b + c.$$

Let us write Δ' for the discriminant of $\psi(X, Y, Z)$, namely

$$a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2.$$

If there is a change of origin for which the equation $\phi(x, y, z) = 0$ is unchanged, the same change of origin must leave the equation $\psi(X, Y, Z) = 0$ unchanged; that is to say, if Δ is zero Δ' is zero. Also when the origin is not changed

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2,$$

therefore

$$\phi(x, y, z) - k(x^2 + y^2 + z^2) = \psi(X, Y, Z) - k(X^2 + Y^2 + Z^2)$$

for all values of k.

98. *Simplification of $\phi(x, y, z)$ by change of axes, when Δ is zero.* If Δ is zero, let the new directions of the axes be chosen so that the line of vertices of the surface $\phi(x, y, z) = 0$, which exists in this case, is the new axis of Z. Then the direction cosines of this line, with reference to the new axes, are $(0, 0, 1)$; and they must satisfy the equations for a line of vertices, namely:

$$a'l + h'm + g'n = 0,$$
$$h'l + b'm + f'n = 0,$$
$$g'l + f'm + c'n = 0.$$

Therefore g', f' and c' are zero, and $\phi(X, Y, Z)$ takes the form

$$a'X^2 + b'Y^2 + 2h'XY.$$

Now we know from conics that, if h' is not zero, the axes of X and Y can be turned, about the axis of Z, into new positions such that this expression is transformed into

$$LX^2 + MY^2,$$

where X, Y, Z are now the coordinates of a point with reference to the new axes thus obtained. Thus we have found axes with reference to which $\phi(x, y, z)$ takes this form, $LX^2 + MY^2$, when its discriminant is zero.

If either L or M is zero, the surface $\phi(x, y, z) = 0$ is a pair of coincident planes, and the new axis of Z may be any line through the origin in this plane. But if L and M are neither of them zero, the new axis of Z has a unique direction; and the surface is either a single straight line or a pair of intersecting planes.

99. *Extension to cases in which Δ is not zero.* The corresponding simplification of $\phi(x, y, z)$, in the general case in which its discriminant is not required to be zero, depends on the fact that there are in all cases real values of k for which the discriminant of

$$\phi(x, y, z) - k(x^2 + y^2 + z^2)$$

is zero. This will now be proved.

The equation to be satisfied is

$$(a-k)(b-k)(c-k)+2fgh \\ -(a-k)f^2-(b-k)g^2-(c-k)h^2=0,$$

or $$k^3-(a+b+c)k^2+(bc+ca+ab-f^2-g^2-h^2)k \\ -(abc+2fgh-af^2-bg^2-ch^2)=0.$$

This cubic equation for k is called the discriminating cubic of $\phi(x, y, z)$. It has at least one real root, because every cubic equation with real coefficients has a real root. Let this root be γ; then new directions can be chosen for the axes, and coefficients L and M found, such that

$$\phi(x, y, z)-\gamma(x^2+y^2+z^2)=LX^2+MY^2,$$

where X, Y, Z are the coordinates of a point referred to the new axes, and γ is zero if Δ is zero. Also

$$x^2+y^2+z^2=X^2+Y^2+Z^2,$$

therefore

$$\phi(x, y, z)=(L+\gamma)X^2+(M+\gamma)Y^2+\gamma Z^2,$$

which may be written

$$\phi(x, y, z)=\alpha X^2+\beta Y^2+\gamma Z^2.$$

The new axes of X, Y, Z, here chosen, are called principal axes of $\phi(x, y, z)$, or of the surface $\phi(x, y, z)=0$.

It is obvious that α and β are roots of the discriminating cubic, as well as γ, because

$$\phi(x, y, z)-\alpha(x^2+y^2+z^2)=(\beta-\alpha)Y^2+(\gamma-\alpha)Z^2,$$

and

$$\phi(x, y, z)-\beta(x^2+y^2+z^2)=(\alpha-\beta)X^2+(\gamma-\beta)Z^2,$$

and the discriminant of the right hand side, and therefore also of the left hand side, of each of these equations is zero. Therefore α, β, γ are the three roots of the cubic if they are unequal.

If however two of them, say β and γ, are equal, a doubt may be felt as to whether the roots of the cubic may not be α, α, γ instead of α, γ, γ. But this doubt can be set at

rest, because the form of the cubic equation shows that the sum of its roots is $a + b + c$, and it has been proved, (§ 97), that the sum of the three coefficients, α, β, γ, is equal to $a + b + c$.

The case of the coefficients α, β and γ being all equal can occur only when f, g and h are all zero; because no transformation of $X^2 + Y^2 + Z^2$ can introduce product terms. And in all cases in which f, g and h are zero

$$\phi(x, y, z) - k(x^2 + y^2 + z^2)$$

takes the form

$$(a - k)x^2 + (b - k)y^2 + (c - k)z^2;$$

and the discriminating cubic is

$$(a - k)(b - k)(c - k) = 0,$$

the three roots of which are a, b, c.

Thus there is no exception to the rule that $\phi(x, y, z)$ can, in all cases, by a change of the directions of the axes if this is necessary, be written in the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2,$$

where α, β, γ are the three roots of its discriminating cubic, a unique set of three real numbers.

100. *Discriminating cubic.* It has been proved here, incidentally, that the discriminating cubic is an equation whose roots are all real. This is a proposition which can be proved independently by the methods of the theory of equations.

It is worth while to notice that the signs of α, β and γ, which may be the most important thing to discover, depend only on the signs of the coefficients in the cubic. For it is a well known rule, for algebraical equations whose roots are all real, that the number of positive roots is equal to the number of changes of sign of the successive terms of the equation. For example, the discriminating cubic of $6yz + 4zx + 2xy$ is

$$k^3 - 14k - 12 = 0.$$

Calculation of the roots of this equation presents no difficulty. But the thing that is seen at a glance is that there is one change of sign. Therefore one, and only one, root is positive; and as there is no zero root, two roots are negative.

101. *Directions of Principal Axes.* Any set of lines which can be adopted as the axes of X, Y and Z, when $\phi(x, y, z)$ is written in the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2,$$

is a set of principal axes of $\phi(x, y, z)$.

To find the direction cosines of these axes, when f, g and h are not all zero, consider first the new axis of Z. If γ is not equal to α or to β, L and M, (§ 99), are neither of them zero. And the new axis of Z, namely that which is adopted for the transformation of

$$\phi(x, y, z) - \gamma(x^2 + y^2 + z^2)$$

into $$LX^2 + MY^2,$$

is the line whose direction cosines, l, m, n, are given uniquely by the equations $A = 0$, $B = 0$, $C = 0$ in § 96, which now take the form

$$(a - \gamma)\, l + hm + gn = 0,$$

$$hl + (b - \gamma)\, m + fn = 0,$$

$$gl + fm + (c - \gamma)\, n = 0.$$

But if γ is equal to one of the other coefficients, say β, and is not equal to α, so that M is zero and L is not zero, the surface

$$\phi(x, y, z) - \gamma(x^2 + y^2 + z^2) = 0$$

is a pair of coincident planes, coinciding with the plane of YZ, and the new axis of Z may be any line through the origin in this plane. Its direction still satisfies the equations written above, but they no longer give a unique direction.

Accordingly we have in this case a single direction for the new axis of X, given by the equations

$$(a-\alpha)\,l + hm + gn = 0,$$

$$hl + (b-\alpha)\,m + fn = 0,$$

$$gl + fm + (c-\alpha)\,n = 0;$$

and the new axes of Y and Z have any directions, at right angles to each other, in a plane at right angles to this new axis of X.

The other axes follow corresponding rules, so these statements deal with the directions of principal axes in all cases. If α, β and γ are all unequal, the equations give a unique set of three directions for the principal axes. If they are all equal any set of three directions at right angles is a set of directions of principal axes.

In an actual transformation it may be convenient, having chosen two sets of direction cosines of new axes, to choose the signs of the third set so as to make the whole transformation rotational, (§ 50).

102. The fact that the equations for the direction cosines of principal axes give directions which are at right angles to one another, for any two principal axes corresponding to unequal roots of the discriminating cubic, can be verified independently as follows.

Consider the axes of Y and Z, and let (l, m, n), (l', m', n') be their direction cosines referred to the original axes. Then we have

$$\beta l = al + hm + gn,$$

$$\beta m = hl + bm + fn,$$

$$\beta n = gl + fm + cn.$$

Therefore $\beta\,(ll' + mm' + nn')$ is equal to

$$all' + bmm' + cnn'$$
$$+ f\,(mn' + m'n) + g\,(nl' + n'l) + h\,(lm' + l'm).$$

Now the symmetry of this expression shows that it is also equal to $\gamma(ll' + mm' + nn')$. Therefore

$$(\beta - \gamma)(ll' + mm' + nn') = 0.$$

And as $\beta - \gamma$ is not zero,

$$ll' + mm' + nn' = 0,$$

which is the condition of perpendicularity.

103. *Invariants.* Let us return now to the general transformation of $\phi(x, y, z)$ into $\psi(X, Y, Z)$ by any given change of the directions of the axes, (§ 97). As α, β, γ are a unique set of numbers, namely the roots of the discriminating cubic of $\phi(x, y, z)$, they are also the roots of the discriminating cubic of $\psi(X, Y, Z)$. Therefore we have, not only

$$a + b + c = a' + b' + c',$$

(§ 97), but also

$$bc + ca + ab - f^2 - g^2 - h^2 = b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2,$$

and

$$abc + 2fgh - af^2 - bg^2 - ch^2 = a'b'c' + 2f'g'h' - af'^2 - bg'^2 - ch'^2.$$

The three functions of the coefficients which are thus shown to be unchanged are called invariants for this transformation. They are, of course, equal respectively to

$$\alpha + \beta + \gamma, \quad \beta\gamma + \gamma\alpha + \alpha\beta, \quad \alpha\beta\gamma.$$

104. *Condition for Equal Roots.* If f, g and h are all zero the roots of the discriminating cubic of $\phi(x, y, z)$ are a, b, c; and it is only in this case that the three roots can be all equal.

Accordingly our problem is to find the condition that two of the roots, α, β, γ, are equal when f, g and h are not all zero.

We have seen that the roots of the discriminating cubic are specified by the equation

$$\phi(x, y, z) = \alpha X^2 + \beta Y^2 + \gamma Z^2,$$

where X, Y, Z are linear functions of x, y, z, subject to the condition

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2.$$

Now these two equations are equivalent to the single equation

$$\begin{aligned}\phi(x, y, z) - \lambda(x^2 + y^2 + z^2)\\ = (\alpha - \lambda) X^2 + (\beta - \lambda) Y^2 + (\gamma - \lambda) Z^2,\end{aligned}$$

assumed to be valid for all values of λ; and this equation shows how the required test for equal roots may be obtained. For it is clear that the right hand side of this equation, regarded as a function of X, Y, Z, is a perfect square if, and not unless, two of the coefficients, $\alpha - \lambda$, $\beta - \lambda$, $\gamma - \lambda$, are zero; and this means that two of the roots must be equal, and λ equal to the repeated root. Now we know, (§ 53), that the right hand side is a perfect square if, and not unless, the left hand side, regarded as a function of x, y, z, is a perfect square. Therefore the condition that two roots are equal is that

$$(a - \lambda) x^2 + (b - \lambda) y^2 + (c - \lambda) z^2 + 2fyz + 2gzx + 2hxy$$

is a perfect square, that is to say can be written in one of the two forms

$$\pm (px + qy + rz)^2;$$

and λ is then equal to the repeated root.

Let us find the condition for this in terms of the six coefficients a, b, c, f, g, h.

If f, g and h are none of them zero, the perfect square must be

$$fgh\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right)^2,$$

in order to provide for the coefficients of yz, zx and xy.

And the coefficients of x^2, y^2 and z^2 must conform to this, that is to say

$$a - \lambda = \frac{gh}{f}, \quad b - \lambda = \frac{hf}{g}, \quad c - \lambda = \frac{fg}{h}.$$

Therefore the required condition is

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h};$$

and each of these three expressions is equal to the repeated root.

If any one of the coefficients f, g and h is zero, we cannot get a perfect square unless two of them are zero, because either p or q or r must be zero. But we have a perfect square if g and h and $a - \lambda$ are zero, provided that

$$(b - \lambda)(c - \lambda) - f^2 = 0.$$

Therefore we have two equal roots if

$$g \text{ and } h \text{ and } (b - a)(c - a) - f^2$$

are all zero; and similarly if

$$h \text{ and } f \text{ and } (c - b)(a - b) - g^2$$

are all zero; or if

$$f \text{ and } g \text{ and } (a - c)(b - c) - h^2$$

are all zero. The repeated root in these cases, respectively, being a, b and c.

In every case in which two of the roots are equal

$$\alpha X^2 + \beta Y^2 + \gamma Z^2,$$

and therefore also $\phi(x, y, z)$, is symmetrical with regard to a straight line, namely one of the principal axes. And as $\phi(x, y, z)$ may be written

$$\pm (px + qy + rz)^2 + \lambda (x^2 + y^2 + z^2),$$

the direction cosines of this axis are proportional to p, q, r, for it must be a line with regard to which

$$px + qy + rz$$

is symmetrical.

If the two equal roots are zero the equation

$$\phi(x, y, z) = 0$$

must represent a pair of coincident planes. Thus if f, g and h are none of them zero the condition for this is

$$a - \frac{gh}{f} = 0, \quad b - \frac{hf}{g} = 0, \quad c - \frac{fg}{h} = 0.$$

Also if any one of the coefficients f, g, h is zero, we have three alternative conditions as before: namely that either a, g, h and $bc - f^2$ are zero; or b, h, f and $ca - g^2$; or c, f, g and $ab - h^2$.

105. *Cone of Revolution.* The same question may be approached by forming the general equation of a cone of revolution with its vertex at the origin.

If l, m, n are direction cosines, and p^2/a^2 is restricted to being not greater than 1, the equation

$$a^2(lx + my + nz)^2 - p^2(x^2 + y^2 + z^2) = 0$$

obviously represents a cone drawn through the points of intersection of the sphere

$$x^2 + y^2 + z^2 = a^2$$

and the pair of parallel planes

$$lx + my + nz = \pm p;$$

and is therefore the required equation. If a^2 and p^2 are replaced by unrestricted coefficients, the equation retains the same symmetry, though it may not represent a real cone.

106. *Cone referred to Principal Axes.* It has been proved that axes can be chosen such that, referred to them, the equation of the surface $\phi(x, y, z) = 0$ is

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0;$$

where α, β, γ are the roots of the discriminating cubic. And the axes thus chosen are called principal axes of the surface.

If the roots are all positive, or all negative, the surface is a single point. If one, and only one, root is zero, the surface is either a single straight line, or else a pair of intersecting planes. If two roots are zero the surface is a pair of coincident planes.

In all other cases the equation can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Therefore this equation comprises every case of a quadric cone with a single vertex. It shows that the sections of any quadric cone, with a single vertex, by planes parallel to one of its principal planes are ellipses. If one of these sections, say the section by a plane $z = k$, is known, we have a very simple specification of the surface.

In § 25 it has been proved that the equation of the tangent plane of this cone at a point (x', y', z') is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} - \frac{z'z}{c^2} = 0.$$

Thus the direction cosines of the normal of the cone at the point (x', y', z') are proportional to

$$\frac{x'}{a^2},\ \frac{y'}{b^2},\ -\frac{z'}{c^2}.$$

107. *Circular Sections.* To find the circular plane sections, the equation of the cone may be written

$$x^2 + y^2 + z^2 = \left(1 + \frac{a^2}{c^2}\right) z^2 - \left(\frac{a^2}{b^2} - 1\right) y^2;$$

and assuming the coordinate axes to be taken so that $a^2 > b^2$, this equation may be written

$$x^2 + y^2 + z^2 = (Az + By)(Az - By).$$

This shows that the sections by all planes parallel to

$$Az + By = 0 \quad \text{or} \quad Az - By = 0$$

are circles. Take for example the plane

$$Az + By = k;$$

this plane cuts the cone in a circle because it meets it at the points of its intersection with the surface

$$x^2 + y^2 + z^2 = k(Az - By),$$

which is a sphere with its centre in the plane of yz.

It is obvious from the shape of a cone with elliptic sections that it has only these two systems of circular sections.

108. *Reciprocal Cone.* Each of the two cones

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad a^2x^2 + b^2y^2 - c^2z^2 = 0,$$

is called the reciprocal of the other one; the relation between them being that the generating lines of each cone are parallel to the normals of the other.

To prove this relation, let (l, m, n) be the direction cosines of a generating line of the first cone, and (λ, μ, ν) the direction cosines of the normal at any point on this line. Then l, m, n are connected by the equation

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0,$$

and λ, μ, ν are proportional to $\frac{l}{a^2}, \frac{m}{b^2}, -\frac{n}{c^2}$; therefore λ, μ, ν satisfy the equation

$$a^2\lambda^2 + b^2\mu^2 - c^2\nu^2 = 0.$$

That is to say, lines drawn from the origin parallel to the normals of the first cone are generators of the second cone. The form of the two equations shows that, when this relation exists, it is a reciprocal one.

The result may be stated in another way, namely that the condition that a plane

$$\lambda x + \mu y + \nu z = 0$$

is a tangent plane of a given cone,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

is

$$a^2\lambda^2 + b^2\mu^2 - c^2\nu^2 = 0.$$

This relation between λ, μ and ν is called the tangential equation of the given cone.

109. Writing the equation of a cone with vertex at the origin

$$al^2 + \beta m^2 + \gamma n^2 = 0,$$

the equation of a plane through the origin may in like manner be written, in terms of direction cosines of lines drawn from the origin,

$$Al + Bm + Cn = 0.$$

And the lines, if any, in which the plane cuts the cone, are found by solving these two simultaneous equations for the ratios $l : m : n$.

The condition that the lines of intersection are at right angles is easily expressed. For if (l_1, m_1, n_1), (l_2, m_2, n_2) are the directions of these lines, the condition is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,$$

so that we are only concerned with the ratios

$$l_1 l_2 : m_1 m_2 : n_1 n_2.$$

Elimination of l between the equations of the cone and plane gives

$$\alpha (Bm + Cn)^2 + (\beta m^2 + \gamma n^2) A^2 = 0,$$

or $$(\alpha B^2 + \beta A^2) m^2 + 2\alpha BCmn + (\gamma A^2 + \alpha C^2) n^2 = 0.$$

Therefore $$\frac{m_1 m_2}{n_1 n_2} = \frac{\gamma A^2 + \alpha C^2}{\alpha B^2 + \beta A^2},$$

and as a corresponding result is obtained by the elimination of m, we get

$$\frac{l_1 l_2}{\beta C^2 + \gamma B^2} = \frac{m_1 m_2}{\gamma A^2 + \alpha C^2} = \frac{n_1 n_2}{\alpha B^2 + \beta A^2}.$$

Therefore the condition that the lines are at right angles is

$$(\beta + \gamma) A^2 + (\gamma + \alpha) B^2 + (\alpha + \beta) C^2 = 0.$$

110. *General equation of Tangent Plane.* We have seen that a cone, $\phi(x, y, z) = 0$, with a single vertex, has at every other point a tangent plane which contains the generator at this point. To find the equation of this plane, draw a straight line,

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

through this point P, (x', y', z'), with direction cosines (l, m, n). The values of ρ for the points at which this line meets the surface are given by the equation

$$\phi(x' + l\rho, \quad y' + m\rho, \quad z' + n\rho) = 0,$$

or $\quad \rho^2\phi(l, m, n) + 2\rho(Al + Bm + Cn) + \phi(x', y', z') = 0,$

where
$$A = ax' + hy' + gz',$$
$$B = hx' + by' + fz',$$
$$C = gx' + fy' + cz'.$$

Now $\phi(x', y', z')$ is zero; and the tangent lines at P, other than the generator, are those for which this equation has two zero roots, that is to say the lines for which

$$Al + Bm + Cn = 0,$$

that is to say the lines at right angles to the direction (A, B, C). And the tangent plane at P is the plane which contains these lines; thus its equation is

$$(ax' + hy' + gz')x + (hx' + by' + fz')y + (gx' + fy' + cz')z = 0.$$

111. *Tangential equation of the cone* $\phi(x, y, z) = 0$. The condition that the plane

$$\lambda x + \mu y + \nu z = 0,$$

is a tangent plane of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

is called the tangential equation of the cone. It is found by eliminating z between these two equations, and expressing

the condition that the resulting quadratic equation for the ratio x/y has equal roots. This quadratic equation is

$$\nu^2 (ax^2 + by^2 + 2hxy) - 2\nu (fy + gz)(\lambda x + \mu g) + c(\lambda x + \mu y)^2 = 0;$$

and the condition for equal roots is

$$(a\nu^2 - 2g\nu\lambda + c\lambda^2)(b\nu^2 - 2f\mu\nu + c\mu^2) - (h\nu^2 - f\nu\lambda - g\mu\nu + c\lambda\mu)^2 = 0.$$

This equation, divided throughout by ν^2, gives the required condition namely

$$(bc - f^2)\lambda^2 + (ca - g^2)\mu^2 + (ab - h^2)\nu^2 + 2(gh - af)\mu\nu + 2(hf - bg)\nu\lambda + 2(fg - ch)\lambda\mu = 0.$$

Therefore this is the tangential equation of the given cone. By making f, g and h zero, it is seen to agree with the previous result.

Let us write this equation $\Phi(\lambda, \mu, \nu) = 0$, or

$$\mathbf{A}\lambda^2 + \mathbf{B}\mu^2 + \mathbf{C}\nu^2 + 2\mathbf{F}\mu\nu + 2\mathbf{G}\nu\lambda + 2\mathbf{H}\lambda\mu = 0.$$

Then $\Phi(x, y, z) = 0$ is the equation of the cone reciprocal to $\phi(x, y, z) = 0$. Thus $\phi(x, y, z)$ and $\Phi(x, y, z)$ have the same principal axes. And this, being only a matter of algebra, is true for functions of this form whether the equations represent real cones or not.

112. Another result obtained by the same procedure is the condition that the cone, $\phi(x, y, z) = 0$, has three generating lines at right angles to one another. Let (x_1, y_1, z_1), (x_2, y_2, z_2) be the directions of the two lines in which the plane

$$\lambda x + \mu y + \nu z = 0$$

cuts the cone. Then the quadratic equation for x/y gives, by the formula for the product of the roots,

$$\frac{x_1 x_2}{y_1 y_2} = \frac{b\nu^2 - 2f\mu\nu + c\mu^2}{a\nu^2 - 2g\nu\lambda + c\lambda^2}.$$

And we have two other similar formulae, therefore

$$\frac{x_1 x_2}{b\nu^2 - 2f\mu\nu + c\mu^2} = \frac{y_1 y_2}{c\lambda^2 - 2g\nu\lambda + a\nu^2} = \frac{z_1 z_2}{a\mu^2 - 2h\lambda\mu + b\lambda^2}.$$

And the condition that these lines are at right angles is

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

or

$$a(\mu^2 + \nu^2) + b(\nu^2 + \lambda^2) + c(\lambda^2 + \mu^2) - 2f\mu\nu - 2g\nu\lambda - 2h\lambda\mu = 0,$$

or $$(a + b + c)(\lambda^2 + \mu^2 + \nu^2) - \Phi(\lambda, \mu, \nu) = 0.$$

But if the cone has three generators at right angles, the plane can be drawn through two of them, and the third is then perpendicular to the plane, that is to say has direction (λ, μ, ν). Therefore $\Phi(\lambda, \mu, \nu) = 0$; so that the condition for three perpendicular generators is

$$a + b + c = 0.$$

And as this condition does not involve (λ, μ, ν), if it is satisfied every generator of the cone is a member of a set of three perpendicular generators.

Referred to any one of these sets of generators as rectangular axes the equation of the cone, being satisfied at all points on each axis, takes the form $fyz + gzx + hxy = 0$. So the invariants make $a + b + c = 0$ necessary.

113. *Examples.*

1. Put each of the following functions of x, y, z into the form $\alpha X^2 + \beta Y^2 + \gamma Z^2$, and find the directions of the new axes of X, Y and Z:

(i) $yz + zx + xy$.

(ii) $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy$.

(iii) $32x^2 + y^2 + z^2 + 6yz - 16zx - 16xy$.

(iv) $(y + z)^2 + (z + x)^2 + (x + y)^2$.

[(i) $-\frac{1}{2}(X^2 + Y^2) + Z^2$, direction of axis of Z $(1, 1, 1)$.

(ii) $12X^2 + 4Y^2 - 3Z^2$, directions $(1, 2, 1)$, $(1, 0, -1)$, $(1, -1, 1)$.

(iii) $36X^2 - 2Y^2$, direction of axis of Z $(1, 2, 2)$.

(iv) $X^2 + Y^2 + 4Z^2$, direction of axis of Z $(1, 1, 1)$.]

2. Find the condition that the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

has three perpendicular tangent planes.

3. Prove that if the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represents a cone of revolution, and f, g, h are none of them zero, the cosine of the semivertical angle is

$$\left(\frac{\lambda fgh}{g^2h^2 + h^2f^2 + f^2g^2}\right)^{\frac{1}{2}}, \text{ where } \lambda = \frac{gh}{f} - a = \frac{hf}{g} - b = \frac{fg}{h} - c;$$

and that, if f and g are zero, it is $\{c/(2c - a - b)\}^{\frac{1}{2}}$. (C.)

4. Prove that if the point (f, g, h) is on the cone $ax^2 + by^2 + cz^2 = 0$, the three other points of the cone whose normals pass through (f, g, h) lie on the plane

$$(a - b)(a - c) fx + (b - c)(b - a) gy + (c - a)(c - b) hz = bcf^2 + cag^2 + abh^2. \qquad \text{(C.)}$$

5. Two cones of revolution have the axis of z for a common generator, their axes meet at the point $(k, 0, 0)$, and their vertices are $(0, 0, c)$ and $(0, 0, c')$. Prove that they intersect in a conic lying in the plane

$$2cc'z = k(c + c')x. \qquad \text{(C.)}$$

6. A straight line moves so as to make with any number of fixed lines the angles $\theta_1, \theta_2, \ldots$, so that $a_1 \cos\theta_1 + a_2 \cos\theta_2 + \ldots$ is constant, the lines all passing through a fixed point, and $a_1, a_2, \ldots$ being given numbers, show that the moving line is on the surface of a cone of revolution.

7. Find the circular sections of the cone

$$x^2 + 2y^2 - 3z^2 + 4yz = 0.$$

8. Prove that the tangent plane at any point of the cone

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0$$

intersects the cone

$$ayz + bzx + cxy = 0$$

in two straight lines at right angles to one another. (S.)

9. The axis of z is a generator of a right circular cone, whose axis has direction cosines (λ, μ, ν); find the other line in which the plane $y = kx$ cuts the cone. (C.)

CHAPTER IX

GENERAL EQUATION OF THE SECOND ORDER

114. The general equation of the second order, which by definition represents any quadric surface, in any position with regard to the coordinate axes, will be written, as in § 19,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0;$$

or

$$F(x, y, z) = 0.$$

This equation has ten coefficients, therefore its signification depends on nine numbers, namely the ratios to one another of these coefficients; and any formula expressing the form or position of the surface, in terms of the coefficients of the equation, must involve only these ratios.

If the coordinates of nine points on a quadric surface are given, substitution of these in the equation $F(x, y, z) = 0$ gives nine linear equations for the ten coefficients, from which the nine ratios of these coefficients to one another can, in general, be found. Therefore, subject to certain exceptions, one and only one quadric surface can be drawn through nine given points; and the complete locus of an equation is determined by its real locus. It is obvious that there are exceptions; for nine points may be so related to one another that the nine linear equations, derived from them, are not independent, and therefore fail to specify the ratios of the coefficients completely. For example, nine points on the curve of intersection of two quadric surfaces cannot specify one of them. Also the real locus of an equation may be too scanty to be capable of supplying real points which will suffice to specify it. This is the case, for example, if the real locus consists only of a single straight line. The equations of the second

order which have no real locus may be regarded as a minority; for it is obvious that any such equation can be brought into the class of equations which have a real locus by an alteration of only one coefficient.

Here the axes may be oblique. With regard to this, see note 1 in the Appendix, § 235.

115. *Plane Sections.* Every plane section of a quadric is a conic. For the plane of xy may be any given plane, and putting $z = 0$ we get

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$$

as the equation, in x and y coordinates, of the section of the surface by this plane. Accordingly the term conic in this proposition must be understood to comprise whatever this equation is capable of representing, whether ellipse, or hyperbola, or parabola, or circle, or pair of straight lines, or single straight line, or single point.

Any section by a plane parallel to this one is obtained by putting $z = k$ in the equation of the quadric. Thus we get as the equation of this section, in x and y coordinates,

$$ax^2 + by^2 + 2hxy + 2u'x + 2v'y + d' = 0,$$

the first three coefficients being the same as before, and the other three being new ones. This shows that any two sections of a quadric by parallel planes are two conics, whose equations, referred to parallel sets of axes in their planes, have the same terms of the second order. This result is sometimes expressed by the statement that any two such sections of a quadric are similar, and similarly situated conics.

If this convenient but loose statement is adopted, attention must be paid to the meaning of it, as expressed by the equations of the two sections. For it requires that a hyperbola, and its conjugate hyperbola, and their asymptotes, should be regarded as three similar and simi-

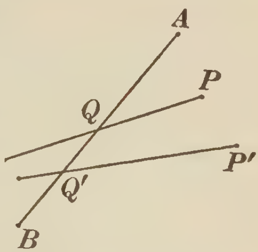
larly situated conics; also a parabola and a straight line parallel to its axis. Moreover when a, b and h are all zero, the two parallel sections are single straight lines not specially related to one another. For example, the sections of the surface

$$2zx - y - z + 1 = 0$$

by the planes $z = 0$ and $z = 1$, are given by the equations $y - 1 = 0$ and $2x - y = 0$ respectively.

116. *Straight Lines on a Quadric.* The fact that plane sections of a quadric surface are conics, in the sense in which this term has been used, gives information about the possible arrangements of straight lines that can be drawn on this surface.

It shows that if any part of a plane section of a quadric is a straight line, the section must be either a pair of straight lines or a single straight line. Let AB be a straight line on a given quadric, and let P be any point on the surface, not on this line. Then the section by the plane ABP is a pair of straight lines, namely AB and PQ, intersecting at Q, it being assumed that they are not parallel. Similarly if P' is another point on the surface, the section by the plane ABP' is the pair of lines AB and $P'Q'$, intersecting at Q'. If Q and Q' do not coincide, PQ and $P'Q'$ cannot intersect, for if they did we should have a plane section containing three straight lines. Thus the surface can be covered with straight lines, which do not intersect each other, but which intersect AB.



Similarly if we start with any one line of this system, say PQ, we can draw another system of lines, of which AB is a member, covering the surface, and intersecting PQ.

But if Q and Q' coincide the surface is a cone. This can be seen by taking the lines QA, QP, QP' for coordinate

axes. This is possible as these lines are not in one plane. The equation of the surface then becomes

$$fyz + gzx + hxy = 0,$$

because it must be satisfied at all points on the axes. And this represents a cone.

In the excluded case of PQ', $P'Q'$...... being all parallel to AB the surface is a cylinder.

It follows from this investigation that every quadric which has a straight line on it is a ruled surface, (§ 23).

117. *Chords*. A straight line which forms a chord of a quadric surface, cutting it in two points, is also a chord of a plane section of the surface, and cannot meet the surface in any third point because it cannot meet a conic in three points. There are some limiting cases. The line may become a tangent of the conic, and therefore also of the quadric. Or the chord may become infinite in length, and cut the conic, and therefore the quadric, in only one point; the line then points in a direction in which the conic, and therefore also the quadric, extends to infinity. Or the line may become an asymptote of the conic, and therefore asymptotic to the quadric.

118. The properties of the surface, $F(x, y, z) = 0$, may be investigated by seeking the points in which a straight line meets it. Draw a straight line, with direction cosines (l, m, n), through a given point P, (x', y', z'), not at infinity. The equations of this line are

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

where ρ is the algebraical distance, measured in the direction (l, m, n), of a point, (x, y, z), on the line, from the point P. The values of ρ for the points in which this line meets the surface are given by the equation

$$F(x' + l\rho, \quad y' + m\rho, \quad z' + n\rho) = 0,$$

or $$K\rho^2 + 2(Al + Bm + Cn)\rho + D = 0,$$

where
$$K = al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm,$$
$$A = ax' + hy' + gz' + u,$$
$$B = hx' + by' + fz' + v,$$
$$C = gx' + fy' + cz' + w,$$
$$D = F(x', y', z') = Ax' + By' + Cz' + E,$$
$$E = ux' + vy' + wz' + d.$$

Thus A, B, C and D depend only on the position of the point P, and K depends only on the direction of the line. And the condition that P is on the surface is $D = 0$.

It will be noticed that $2A$, $2B$, $2C$ are the values at P of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ respectively. The notation of § 165 may be used if it is preferred.

The line lies wholly on the surface if, and not unless, K, $Al + Bm + Cn$ and D are all zero; for this is the condition that the equation

$$K\rho^2 + 2(Al + Bm + Cn)\rho + D = 0$$

is satisfied for all values of ρ.

Except in this case, we have a quadratic equation for ρ, it being assumed that we may call the equation a quadratic in the limiting cases in which one or both of its roots are infinite. If the roots of this quadratic are imaginary the line does not meet the surface. If they are real they may be equal, or one or both of them may be infinite. The surface extends to infinity, or does not do so, according as there are, or are not, directions (l, m, n) for which $K = 0$; for a line drawn in any such direction either lies wholly on the surface, or else meets it at infinity. The whole set of lines through P for which $K = 0$, if there are any, form a cone which may be called the cone K. They point in the directions in which the surface extends to infinity, the same for all positions of P. If P is a point for which A, B, C and E, (and therefore D), are all zero, the quadratic equation becomes $K\rho^2 = 0$; so

in this case the surface is the cone K if this cone exists, but is the single point P if this cone does not exist.

119. *Condition for a Cone.* It must however be noticed that the general formula obtained by the elimination of x', y' and z' between the equations $A = 0$, $B = 0$, $C = 0$, $E = 0$, namely

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} = 0,$$

which is the condition, in terms of the ten coefficients, that these four equations are compatible, is not the condition that the surface is a single point or a cone, in the sense in which the term cone is used here. It is the condition that the surface is either a single point or a cone or a cylinder. The surface is either a single point or a cone if the four equations can be satisfied simultaneously by finite values of x', y' and z'; and it is a cylinder which is not also a cone, if they can be satisfied only in a limiting case of one or more of these numbers tending to infinity.

The determinant written above is called the discriminant of $F(x, y, z)$. It may be written

$$-\{(bc - f^2)u^2 + (ca - g^2)v^2 + (ab - h^2)w^2$$
$$+ 2(gh - af)vw + 2(hf - bg)wu + 2(fg - ch)uv\} + \Delta d,$$

or
$$-\Phi(u, v, w) + \Delta d,$$

where Φ has the same meaning as in § 111, and Δ is, as in § 96, the discriminant of

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

For further investigation of the distinction between a cone and a cylinder, see note 2 in the Appendix, § 235.

120. *Centre of quadric.* If P is a point for which A, B and C are all zero it is called the centre, or a centre, of

the surface. The quadratic equation then becomes

$$K\rho^2 + D = 0$$

for all directions of the line. This shows that, if the surface is a cone, a centre is a vertex of the cone; and that, if the surface is a single point, the centre is that point; and that, in all other cases, a centre is a point which bisects all chords of the surface drawn through it. Thus we have a definition of a centre which is independent of the choice of coordinate axes.

If the origin is a centre, u, v and w are zero, so that the equation of the surface is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0,$$

or, using the notation of § 95, $\phi(x, y, z) + d = 0$.

The definition of a centre shows that the equations which give the coordinates of a centre are

$$\begin{aligned} ax + hy + gz + u &= 0, \\ hx + by + fz + v &= 0, \\ gx + fy + cz + w &= 0. \end{aligned}$$

In general these equations represent three planes, and a point in which they intersect is a centre. Therefore in general a quadric surface has a unique centre, and all other cases may not improperly be called exceptional. There is a whole line of centres if the equations represent planes which have a common line of intersection; or if all the four coefficients of one of the equations are zero, and the other two equations represent intersecting planes. There is a whole plane of centres if the three equations represent coincident planes; or if two of them represent coincident planes, and all the coefficients of the third are zero; or if one equation represents a plane, and all the coefficients of the other two are zero. In all other cases the surface has no centre, because this set of equations cannot be satisfied by finite values of the coordinates.

The quadric surfaces which have a centre, whether unique or not, are called central quadrics. And if the

complete classification of quadric surfaces, (§§ 125–7), is referred to, it will be seen that the surfaces which have no centre are the paraboloids and the parabolic cylinder. These surfaces may be interpreted as the limiting cases of central quadrics, when the intersection of planes by which the centre is found passes to infinity. It is obvious that there are such cases. The case in which the intersection passes to infinity in a definite direction is that which gives a paraboloid; the direction cosines of this direction being given, as in § 34, by

$$al + hm + gn = 0,$$
$$hl + bm + fn = 0,$$
$$gl + fm + cn = 0.$$

This occurs when the equations represent three planes intersecting in parallel lines; or when two of them represent intersecting planes, and the third represents the limiting case of a plane at infinity, the coefficients of x, y and z being zero and the fourth term not zero. The well-known properties of a parabola show that a parabolic cylinder is obtained from a line of intersections passing to infinity.

The equations for the centre show that for a cone, when the origin is a vertex, u, v, w and d are all zero.

With regard to oblique axes, see § 235, note 1.

121. *Tangent Plane.* Let us now proceed with the interpretation of the quadratic equation

$$K\rho^2 + 2\rho(Al + Bm + Cn) + D = 0$$

for the ordinary case in which the point P, (x', y', z'), is not a centre; so that A, B and C are not all zero, and consequently there is a signless direction (A, B, C), and a plane through P at right angles to this direction.

If P is on the surface, this plane is the tangent plane of the surface at the point P. To prove this let us find the tangent lines at P, a point for which D is zero.

The lines through P which lie on the surface are tangent lines; and their directions are given by

$$K = 0, \quad Al + Bm + Cn = 0.$$

Therefore these lines, if they exist, are at right angles to the direction (A, B, C).

The other tangent lines at P are the tangents of sections of the surface by planes through P. A tangent is found by drawing a chord, PQ, of one of these sections, and taking its limiting position when Q tends to coincidence with P. If (l, m, n) are the direction cosines of PQ, the length of the chord is the positive value of

$$\pm \frac{2}{K}(Al + Bm + Cn).$$

Therefore the limiting position of the chord is one in which

$$Al + Bm + Cn = 0,$$

that is to say is at right angles to (A, B, C); for the case in which K is at the same time zero is excluded.

Thus at every point, (x', y', z'), on a quadric surface, for which A, B and C are not all zero, there is a tangent plane, whose equation is

$$A(x - x') + B(y - y') + C(z - z') = 0,$$

and a normal whose direction is (A, B, C).

122. *Polar Plane.* The plane

$$A(x - x') + B(y - y') + C(z - z') + D = 0$$

is called the polar plane of a point P, (x', y', z'), with regard to the surface, and P is called the pole of the plane. If the point is on the surface, D is zero, and the polar plane is the tangent plane.

If P is not on the surface, let us draw all the straight lines through P which meet the surface in two points. And let

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho$$

be the equations of one of these lines, meeting the surface

at the points for which ρ has the values ρ_1, ρ_2. And let Q, (x'', y'', z''), be the point on this line given by

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}.$$

Then the quadratic equation shows that this is equivalent to

$$\rho = -\frac{D}{Al + Bm + Cn};$$

and $l\rho = x''$, $m\rho = y''$, $n\rho = z''$, therefore

$$A(x'' - x') + B(y'' - y') + C(z'' - z') = -D.$$

This shows that the points, Q, chosen in this way, for different directions of the line through P, are all of them on the plane

$$A(x - x') + B(y - y') + C(z - z') + D = 0,$$

and thus specify the polar plane of P in a way which is independent of the choice of coordinate axes.

This is the same specification as that found for the sphere, (§ 63), and for the ellipsoid, (§ 79); and it may be adopted as the definition of a polar plane for all quadrics.

As P approaches a centre its polar plane passes to infinity, except when the surface is a cone or a single point, in which case it ceases to have any meaning. When the point P is not on the surface, its distance from its polar plane is the positive value of $\pm D(A^2 + B^2 + C^2)^{-\frac{1}{2}}$. For a cone, every polar plane passes through the vertex.

The formula for D, (§ 118), shows that the equation of the polar plane of P may be written

$$Ax + By + Cz + ux' + vy' + wz' + d = 0;$$

and substituting for A, B and C their values, (§ 118), the equation becomes

$$ax'x + by'y + cz'z + f(z'y + y'z) + g(x'z + z'x) + h(y'x + x'y)$$
$$+ u(x + x') + v(y + y') + w(z + z') + d = 0.$$

This equation involves x, y, z and x', y', z' symmetrically; therefore the condition that a point (x_1, y_1, z_1) is on the polar plane of a point (x_2, y_2, z_2) is the same as the con-

dition that (x_2, y_2, z_2) is on the polar plane of (x_1, y_1, z_1). Draw all the tangent planes which pass through some given point P. Their poles are their points of contact, and they must all lie on the polar plane of P. This shows that if P is a point from which tangent lines to the surface can be drawn, all their points of contact are in the polar plane of P, which may therefore be called the plane of contact of a cone drawn, with vertex P, to envelop the surface.

Other pole and polar properties of any quadric are either the same as those for a sphere or an ellipsoid, or analogous to them.

Let P and Q be two points whose polar planes with regard to the quadric intersect in a line RS. Then the polar planes of all points on the line RS contain the line PQ, and the polar planes of all points on the line PQ contain the line RS. Two lines which have this reciprocal relation are each of them called the polar line of the other. If PQ is a chord of the surface, the polar line of this chord is the line of intersection of the tangent planes at P and Q. The normals at P and Q intersect when PQ is at right angles to its polar line, as in the case of a sphere, (§ 65).

The condition that the quadratic equation for ρ, § 118, has equal roots is

$$(Al + Bm + Cn)^2 - KD = 0.$$

So this is the equation, in terms of the direction cosines of its generating lines, of a cone drawn, with vertex P, to envelop the quadric, if this cone exists. And by substituting $x - x'$, $y - y'$, $z - z'$ for l, m, n respectively, we get the usual equation of this cone, without exclusion of the vertex.

123. *Diametral Planes and Conjugate Diameters.* The quadratic equation for ρ shows that the condition that a chord of the surface, drawn through P, with direction cosines (l, m, n), is bisected at this point is

$$Al + Bm + Cn = 0.$$

The following propositions are interpretations of this:

(i) If P is a centre of the surface, it bisects every chord through it, because A, B and C are zero.

(ii) If P is not a centre of the surface, the chords through it which it bisects are those which are in the plane through P at right angles to the direction (A, B, C). Therefore, if this plane cuts the surface, P is the centre of the section; and if this plane does not cut the surface, P does not bisect any chords.

(iii) Therefore the centres of the sections by a given set of parallel planes, whose direction cosines are (λ, μ, ν), are the points for which A, B, C are proportional to λ, μ, ν. Therefore their coordinates satisfy the equations

$$\frac{1}{\lambda}(ax + hy + gz + u) = \frac{1}{\mu}(hx + by + fz + v)$$
$$= \frac{1}{\nu}(gx + fy + cz + w).$$

Therefore they are on a straight line, which passes through a centre of the surface, if there is a centre, and points in the direction in which the centre passes to infinity when the quadric is a paraboloid.

(iv) Draw a set of parallel chords with direction cosines (l, m, n). The coordinates of their middle points satisfy the equation

$$(ax + hy + gz + u)\,l + (hx + by + fz + v)\,m$$
$$+ (gx + fy + cz + w)\,n = 0,$$

or

$$(al + hm + gn)\,x + (hl + bm + fn)\,y + (gl + fm + cn)\,z$$
$$+ ul + vm + wn = 0.$$

This equation represents a plane which is called the diametral plane of the parallel chords, and is said to be conjugate to their direction. It passes through a centre of the surface if there is a centre. The direction of the chords is also said to be conjugate to the diametral plane.

124. *Principal Planes.* A diametral plane which is at right angles to the chords which it bisects is called a principal plane of the surface. Thus the direction cosines, l, m, n, of a principal plane, assuming it to exist, are given by the equations

$$\begin{aligned} al + hm + gn &= kl, \\ hl + bm + fn &= km, \\ gl + fm + cn &= kn. \end{aligned}$$

Now these are the same equations as those which give the direction cosines of the principal axes of

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

which we will write, as before, $\phi(x, y, z)$. Therefore the direction cosines of the principal planes of the quadric are those of the principal axes of $\phi(x, y, z)$.

Elimination of l, m, n between the three equations gives

$$(a-k)(b-k)(c-k) + 2fgh \\ - (a-k)f^2 - (b-k)g^2 - (c-k)h^2 = 0,$$

which is the discriminating cubic of $\phi(x, y, z)$. Thus k is one of the roots, α, β, γ, of this cubic. And the discussion of the directions obtained, (§ 101), need not be repeated here. In the normal case, in which there is a centre and the roots are all unequal and not zero, there are three principal planes at right angles to one another.

In the case of a paraboloid the symmetry of the surface shows that there are two principal planes, and that they pass through the vertex; and that the tangent plane at the vertex is at right angles to them.

125. *Classification of Quadrics.* A complete classification of quadric surfaces will now be obtained by examination of the simplest forms to which the equation

$$F(x, y, z) = 0$$

can be reduced by a change of coordinate axes. By making the substitutions, (§ 50), for a change of the directions of the

axes, taking new axes parallel to a set of principal axes of $\phi(x, y, z)$, the equation can always be put into the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2u'x + 2v'y + 2w'z + d = 0.$$

Here u', v', w' are new coefficients arising from the change of axes, and d remains unchanged, and α, β, γ are the roots of the discriminating cubic of $\phi(x, y, z)$, (§ 99).

Consider first the case in which α, β and γ are none of them zero. This is the common case, in which the only stipulation is that

$$abc + 2fgh - af^2 - bg^2 - ch^2,$$

which is denoted by Δ, is not to be zero. Shift the origin to the point $(-u'/\alpha, -v'/\beta, -w'/\gamma)$. The equation then becomes

$$\alpha x^2 + \beta y^2 + \gamma z^2 + d' = 0,$$

where

$$d' = u'^2/\alpha + v'^2/\beta + w'^2/\gamma + d.$$

If d' is zero the real locus of this equation is either a single point, namely the new origin, or else a cone with elliptic sections and with the new origin for vertex. If d' is not zero, divide by $-d'$, the equation then takes the form

$$\alpha' x^2 + \beta' y^2 + \gamma' z^2 = 1.$$

If α', β' and γ' are all negative this equation has no real locus. Otherwise it represents, according to the signs of α', β' and γ', either an ellipsoid (which may be a sphere), or a hyperboloid of one sheet, or a hyperboloid of two sheets, see § 89. Thus we see that each of the surfaces which the equation can represent, if Δ is not zero, has a unique centre, namely the new origin.

The equations which give the coordinates of a centre, referred to the original axes, namely

$$ax + hy + gz + u = 0,$$

$$hx + by + fz + v = 0,$$

$$gx + fy + cz + w = 0,$$

show that the quadrics which have a unique centre are those for which Δ is not zero; and that those for which

Δ is zero have either no centre, or a line of centres, or a plane of centres. In the present case, Δ not being zero, these equations give the coordinates, $(\bar{x}, \bar{y}, \bar{z})$, of the centre, referred to the original axes. And

$$F(x, y, z) = \alpha X^2 + \beta Y^2 + \gamma Z^2 + d',$$

where (X, Y, Z) are the coordinates, referred to the new axes, of a point whose coordinates referred to the original axes are (x, y, z); therefore

$$d' = F(\bar{x}, \bar{y}, \bar{z}).$$

126. To deal with the cases in which $\Delta = 0$, consider first those in which one, and only one, of the roots of the discriminating cubic, say γ, is zero; so that

$$bc + ca + ab - f^2 - g^2 - h^2$$

is not zero.

The equation can then be written

$$\alpha x^2 + \beta y^2 + 2u'x + 2v'y + 2w'z + d = 0.$$

Shift the origin to the point $(-u'/\alpha, -v'/\beta, 0)$. The equation then becomes

$$\alpha x^2 + \beta y^2 + 2w'z + d'' = 0,$$

where $$d'' = u'^2/\alpha + v'^2/\beta + d.$$

If w' is not zero, shift the origin again to the point

$$(0, 0, -d''/2w').$$

The equation then becomes

$$\alpha x^2 + \beta y^2 + 2w'z = 0,$$

which represents a paraboloid, elliptic or hyperbolic according as α and β have the same or opposite signs, (§§ 89 and 91). The new origin is the vertex of the surface, and the new planes of yz and zx are the principal planes.

But if w' is zero, the equation is

$$\alpha x^2 + \beta y^2 + d'' = 0.$$

If α, β and d'' are all positive or all negative, this equation has no real locus. Otherwise it represents a cylinder with

a line of centres, namely the new axis of z. This is either an elliptic or a hyperbolic cylinder, or a pair of intersecting planes, or a single straight line.

The two cases of the surface being a paraboloid or a cylinder can be distinguished, in terms of the ten coefficients of $F(x, y, z)$, by the fact that the cylinder satisfies, and the paraboloid does not satisfy, the condition, (§ 119), for a quadric being a point or cone or cylinder; or by using the formula given in § 235, note 2.

Let us now find for the paraboloid the coordinates of the vertex with reference to the original axes, and the value of w'. The direction cosines, (l, m, n), of the axis of the surface are given, (§ 120), by the equations

$$al + hm + gn = 0,$$

$$hl + bm + fn = 0,$$

$$gl + fm + cn = 0.$$

Thus $$l : m : n :: bc - f^2 : fg - ch : hf - bg.$$

And in order that the cosines l, m, n may specify a single direction, let their signs be chosen so that $ul + vm + wn$ is positive. It will be found below that this expression cannot be zero. Every straight line in this direction meets the surface in one and only one point; and the axis of the surface is the line in this direction at every point of which the direction (A, B, C) is the same as (l, m, n), see § 123, (iii). Therefore the equations of the axis are

$$ax + hy + gz + u = kl,$$

$$hx + by + fz + v = km,$$

$$gx + fy + cz + w = kn.$$

Multiply these equations respectively by l, m and n and add; then the coefficients of x, y and z are

$$al + hm + gn, \quad hl + bm + fn, \quad gl + fm + cn,$$

and are therefore zero, therefore

$$ul + vm + wn = k(l^2 + m^2 + n^2) = k.$$

Adopting the value thus found for k, any two of the three equations are equations of the axis.

The coordinates of the vertex can now be found, for it is the point in which the axis meets the surface $F(x, y, z) = 0$. This equation can be written

$$(ax + hy + gz + u)\,x + (hx + by + fz + v)\,y$$
$$+ (gx + fy + cz + w)\,z + ux + vy + wz + d = 0.$$

Therefore the coordinates of the vertex are given by the equation

$$k\,(lx + my + nz) + ux + vy + wz + d = 0,$$

combined with any two of the three equations of planes through the axis.

To calculate w' we have to pick out the coefficient of $2Z$ when $F(x, y, z)$ is transformed into $\alpha X^2 + \beta Y^2 + 2w'Z$ by the substitutions

$$x = l_1X + l_2Y + lZ + x',$$
$$y = m_1X + m_2Y + mZ + y',$$
$$z = n_1X + n_2Y + nZ + z',$$

where x', y', z' are written for the coordinates of the vertex. This gives

$$w' = l\,(ax' + hy' + gz' + u) + m\,(hx' + by' + fz' + v)$$
$$+ n\,(gx' + fy' + cz' + w) = (l^2 + m^2 + n^2)\,k = k.$$

For the cylinder $d'' = F(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x}, \bar{y}, \bar{z}$ are the coordinates of a centre referred to the original axes, as in § 125. And the line of centres is found from the equations for a centre, (§ 120).

127. The only case that remains is that in which two roots of the discriminating cubic are zero, the condition for which is that $bc + ca + ab - f^2 - g^2 - h^2$ is zero as well as Δ.

If α is the root which is not zero the equation can be written

$$\alpha x^2 + 2u'x + 2v'y + 2w'z + d = 0.$$

By turning the axes of y and z through a suitable angle about the axis of x, an operation which affects only the third and fourth terms, this can be put into the form

$$\alpha x^2 + 2u'x + 2v''y + d = 0;$$

and if the origin is shifted to the point $(-u'/\alpha, 0, 0)$, it becomes

$$\alpha x^2 + 2v''y + d'' = 0.$$

If v'' is not zero this represents a parabolic cylinder; and by a change of origin the equation can finally be written $\alpha x^2 + 2v''y = 0$. If v'' is zero it represents a pair of parallel or coincident planes if there is a real locus; the equation being $\alpha x^2 + d'' = 0$.

The equations of the new coordinate planes with reference to the original axes, and the values of v'' and d'' can be found as follows. Two roots of the cubic being zero means, (§ 104), that $F(x, y, z) = 0$ can be written in one of the two forms

$$\pm (\lambda x + \mu y + \nu z)^2 + 2ux + 2vy + 2wz + d = 0.$$

In the particular case in which λ, μ, ν are proportional to u, v, w, this equation is a quadratic equation for

$$ux + vy + wz,$$

the solution of which, if real, gives two parallel or coincident planes at right angles to the direction (u, v, w). Accordingly, if r, s are the roots of this quadratic equation, we get in the case of a pair of planes

$$\frac{d''}{\alpha} = -\frac{1}{4}\frac{(r-s)^2}{u^2+v^2+w^2}.$$

In all other cases the following procedure, giving a parabolic cylinder, succeeds. Taking the given equation to be

$$(\lambda x + \mu y + \nu z)^2 + 2ux + 2vy + 2wz + d = 0,$$

let this be written

$$(\lambda x + \mu y + \nu z + k)^2 + 2(u - k\lambda)x + 2(v - k\mu)y + 2(w - k\nu)z + d - k^2 = 0,$$

and let k be chosen so that

$$\lambda(u - k\lambda) + \mu(v - k\mu) + \nu(w - k\nu) = 0.$$

Then the equations

$$\lambda x + \mu y + \nu z + k = 0,$$

$$2(u - k\lambda)x + 2(v - k\mu)y + 2(w - k\nu)z + d - k^2 = 0$$

represent two planes at right angles, and if X, Y are the perpendicular distances of a point (x, y, z) from these two planes respectively, the equation of the surface is

$$X^2 + KY = 0,$$

where $$\frac{1}{4}K^2 = \frac{(u - k\lambda)^2 + (v - k\mu)^2 + (w - k\nu)^2}{(\lambda^2 + \mu^2 + \nu^2)^2}.$$

Accordingly these two planes are the new planes of yz and zx in the final new equation of a parabolic cylinder, and $2v''/a$ is equal to K. The first plane is the plane of symmetry of the surface, and the second plane is the tangent plane at right angles to this.

If the given equation of the surface has the alternative form, with a minus sign, the result is the same except for the signs of u, v, w and d being reversed.

128. If all the roots of the discriminating cubic are zero the surface is a single plane, which may be interpreted as a pair of planes one of which has passed to infinity.

The classification of quadric surfaces, defined by the general equation of the second order, is now complete. It will be seen that we have a number of special cases, each of them marked by some special relation or relations between the coefficients. But that there is one normal case marked only by the absence of all these special relations, namely that in which the surface, if real, is either an ellipsoid, or a hyperboloid of one sheet, or a hyperboloid of two sheets.

129. *Surfaces of Revolution.* A subordinate question remains however, namely the condition that the surface $F(x, y, z) = 0$ is a surface of revolution.

We have seen, (§ 85), that a quadric surface is a surface of revolution if, and not unless, coordinate axes can be chosen such that, referred to them, the equation of the surface is

$$\lambda (x^2 + y^2) = Az^2 + Bz + C.$$

Therefore a necessary condition for the surface,

$$F(x, y, z) = 0,$$

being a surface of revolution is that two roots of the discriminating cubic are equal. The condition for this has been found, in terms of the coefficients, in the previous chapter, (§ 104). If this condition is satisfied the equation of the surface can be written

$$a (x^2 + y^2) + \gamma z^2 + 2u'x + 2v'y + 2w'z + d = 0;$$

and if a is not zero, a shift of the origin to the point $(-u'/a, -v'/a, 0)$ will bring this into the form

$$a (x^2 + y^2) + \gamma z^2 + 2w'z + d' = 0,$$

so that no further condition is needed. But if the equal roots are zero a further condition is needed. In this case the surface is either a parabolic cylinder, and therefore not a surface of revolution, or else a pair of parallel or coincident planes. Accordingly the further condition which is needed is that which secures that the surface is a pair of parallel or coincident planes. This has been given in § 127.

130. *Examples.*

1. Show that if two quadrics have a common plane section, their other points of intersection are in one plane. [See § 115.]

2. Prove that three straight lines, no two of which are coplanar, specify a single quadric on which they lie. [See § 43.]

3. Investigate the following surfaces, giving the simplest forms of their equations, referred to new axes of X, Y, Z:

i. $x^2 + y^2 - z^2 + 2yz + 2zx - 2xy + 2x + 2y + 2z - 1 = 0$,

ii. $2x^2 - 6y^2 - 3z^2 - 11yz + 5zx + xy + x + y - 2z + 1 = 0$,

iii. $8x^2 + 5y^2 + 5z^2 - 8yz - 4zx - 4xy + 3x + 6y + 6z = 0$,

iv. $x^2 + y^2 + z^2 + 2yz + 4zx + 2xy + 6x + 8y - 1 = 0$,

v. $4x^2 + y^2 + 4z^2 - 4yz + 8zx - 4xy + 2x - 4y + 5z + 1 = 0$,

vi. $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$,

vii. $x^2 + 2y^2 + 6z^2 + 7yz - 5zx - 3xy + 3x - 4y - 7z = 0$,

viii. $3z^2 - 4yz - 4zx + 2xy - 4x - 4y + 6z + 4 = 0$,

ix. $5x^2 + 6y^2 + 7z^2 + 4yz - 4xy - 1 = 0$,

x. $x^2 + y^2 + z^2 + 2yz + 4zx + 4xy + 2x - 2y - 2z - 2 = 0$,

xi. $9x^2 - 6y^2 - 7z^2 + 12yz + 18zx + 4xy - 1 = 0$.

[i. $2X^2 + Y^2 - 2Z^2 = 4$; directions of axes $(1, -1, 0)$, $(1, 1, 1)$, $(1, 1, -2)$.

ii. Hyp. paraboloid $\sqrt{3}\,(21X^2 - 7Y^2) = 8Z$; direction of axis of Z $(1, 1, -1)$; coordinates of vertex $(-\frac{80}{441}, -\frac{128}{441}, \frac{119}{441})$.

iii. Paraboloid of revolution $X^2 + Y^2 + Z = 0$; origin unchanged.

iv. $(2 + \sqrt{3})\,X^2 + (2 - \sqrt{3})\,Y^2 - Z^2 = 25$; coordinates of centre $(4, -9, 1)$.

v. Parabolic cylinder $3Y^2 = X$; equations of the new planes of YZ and ZX are $2x + 2y - z - 1 = 0$, and $2x - y + 2z + 1 = 0$.

vi. Elliptic cylinder $2X^2 + Y^2 = 1$; axes $(1, 1, 2)$, $(1, -1, 0)$, $(1, 1, -1)$.

vii. Hyp. cylinder $(\sqrt{84} + 9)\,X^2 - (\sqrt{84} - 9)\,Y^2 = 4$; axis of Z is given by the equations $x = -y + 1 = z$.

viii. Hyperboloid of revolution $X^2 + Y^2 - 5Z^2 = 1$.

ix. Ellipsoid $9X^2 + 6Y^2 + 3Z^2 = 1$.

x. Pair of intersecting planes.

xi. Hyp. of two sheets $4X^2 - 14Y^2 + 14Z^2 = -1$; axes $(-1, 2, 1)$, $(4, 1, 2)$, $(1, 2, -3)$.]

4. Show that the most general quadric surface which has the lines $x = 0, y = 0$; $x = 0, z = c$; $y = 0, z = -c$ as generators is

$$fy\,(z - c) + gx\,(z + c) + hxy = 0,$$

where f, g, h are arbitrary constants. (S.)

5. Prove that the vertices of all paraboloids which pass through the two lines $y = 0$, $z = h$; $x = 0$, $z = -h$; lie on the surface

$$x^2 (z - h)^3 + y^2 (z + h)^3 = 0. \qquad \text{(C.)}$$

[They can all be obtained by varying the generator in the plane of xy.]

6. Planes are drawn through a given point M, so as to intersect the cone $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$. Show that the centres of the sections so formed are on a quadric having its centre at the middle point of OM, O being the origin. (C.)

[Note that this is not changed by homogeneous strain.]

7. Find the equation of the enveloping cylinder of the quadric $F(x, y, z) = 0$, (§ 114), whose axis has a given direction. (S.)

8. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$ represents an ellipsoid of revolution; where f, g and h are none of them zero, and

$$\lambda = a - gh/f = b - hf/g = c - fg/h.$$

Show that the area of the generating ellipse is

$$\pi \{(a + b + c)\lambda - 2\lambda^2\}^{-\frac{1}{2}}. \qquad \text{(C.)}$$

9. Show that the principal planes of the quadric

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

are given by the equation

$$\begin{vmatrix} \mathbf{A}x + \mathbf{H}y + \mathbf{G}z, & \mathbf{H}x + \mathbf{B}y + \mathbf{F}z, & \mathbf{G}x + \mathbf{F}y + \mathbf{C}z \\ ax + hy + gz, & hx + by + fz, & gx + fy + cz \\ x \quad , & y \quad , & z \end{vmatrix} = 0,$$

A, **B**, **C**, **F**, **G**, **H** being the same as in § 111. (C.)

10. Obtain the condition that the quadric $F(x, y, z) = 0$, (§ 114), is a real ruled surface. (C.)

11. Find the condition that the enveloping cone of the quadric

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

with vertex (p, q, r), is a cone of revolution. (C.)

12. Show that the straight lines which lie in a given plane, and which meet a given quadric in two points, the normals at which intersect, touch a parabola. (S.)

CHAPTER X

SPECIAL FORMS OF THE EQUATION OF A QUADRIC

131. *Central Quadric.* The equation of a central quadric with a unique centre, other than a cone or a single point, referred to its principal axes as coordinate axes, is

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

where α, β, γ may be any set of three numbers, which are none of them zero, and are not all negative. The surface is either an ellipsoid, or a hyperboloid of one sheet, or a hyperboloid of two sheets, (§ 89).

The general form of the equation of this surface, referred to the centre as origin, is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

or

$$\phi(x, y, z) = 1,$$

the coefficients in this equation being any set of six numbers, satisfying the condition that the roots, α, β, γ, of the discriminating cubic of $\phi(x, y, z)$, are none of them zero, and are not all negative. If the signs of the successive terms of the cubic are known, the signs of these roots are known, (§ 100).

132. If the surface $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is an ellipsoid, the equation

$$\alpha x^2 + \beta y^2 + \gamma z^2 = -1$$

has no real locus. But if the surface is a hyperboloid, this equation represents the conjugate hyperboloid, and

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0$$

is the equation of the asymptotic cone common to these two surfaces.

This shows that if each of the equations

$$\phi(x, y, z) = 1 \quad \text{and} \quad \phi(x, y, z) = -1$$

has a real locus, they represent a pair of conjugate hyperboloids, and $\phi(x, y, z) = 0$ is the equation of their common asymptotic cone. Now the sections of these three surfaces by a plane, $z = k$, are found by writing k for z in each of these equations; so that we get three equations in x and y, in which the terms of the second order are the same, namely

$$ax^2 + by^2 + 2hxy,$$

and do not involve k. This shows that all the sections, of all the three surfaces, by any given set of parallel planes, are conics whose equations in x and y coordinates have the same terms of the second order. So that if one is a circle, they are all circles; if one is an ellipse, they are all similar and similarly situated ellipses; if one is a hyperbola, each of them is a hyperbola, either similar to this one and similarly situated, or similar to its conjugate and similarly situated; if one is a parabola they are all parabolas, similarly situated. A pair of intersecting straight lines is here reckoned as a hyperbola, and a pair of parallel lines as a parabola.

133. Some of the properties of an ellipsoid, which in a previous chapter have been derived from properties of a sphere, are particular cases of properties of the more general central quadric

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1.$$

The values of ρ for the points in which the straight line

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

drawn through a point P, (x', y', z'), with direction cosines (l, m, n), meets this surface, are given by the equation

$$\alpha(x' + l\rho)^2 + \beta(y' + m\rho)^2 + \gamma(z' + n\rho)^2 = 1,$$

or $$K\rho^2 + 2(\alpha x'l + \beta y'm + \gamma z'n)\rho + D = 0;$$

where $$K = \alpha l^2 + \beta m^2 + \gamma n^2,$$
and $$D = \alpha x'^2 + \beta y'^2 + \gamma z'^2 - 1.$$

The lines, if any, through P, whose directions satisfy the equation $K = 0$, form a cone which will be called the cone K. And $D = 0$ is the condition that the point P is on the surface.

If K, $\alpha x'l + \beta y'm + \gamma z'n$ and D are all zero, the line lies wholly on the surface. Otherwise we have a quadratic equation for ρ, as in § 118.

For the cone $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$ we have corresponding equations with $\alpha x'^2 + \beta y'^2 + \gamma z'^2$ for the value of D.

134. *Conjugate Planes and Lines.* If P is on the plane
$$\alpha l x + \beta m y + \gamma n z = 0,$$
the quadratic equation is $K\rho^2 + D = 0$, and its roots, if real, and not zero, are equal with opposite signs. Therefore parallel chords of the surface, with direction cosines (l, m, n), are all bisected by this plane. And the plane is called the diametral plane of these chords, (§ 123).

Consider straight lines drawn through the centre of the surface, each of them specified by its direction cosines. Choose any line (l_1, m_1, n_1); also any line, (l_2, m_2, n_2), in the diametral plane of chords parallel to (l_1, m_1, n_1). Then the equation of this plane gives
$$\alpha l_1 l_2 + \beta m_1 m_2 + \gamma n_1 n_2 = 0.$$
And the symmetry of this equation shows that the line (l_1, m_1, n_1) is in the diametral plane of chords parallel to (l_2, m_2, n_2). Let (l_3, m_3, n_3) be the direction cosines of the line of intersection of the two diametral planes here specified. Then
$$\alpha l_2 l_3 + \beta m_2 m_3 + \gamma n_2 n_3 = 0,$$
and $$\alpha l_3 l_1 + \beta m_3 m_1 + \gamma n_3 n_1 = 0.$$
Therefore the plane containing the two lines, (l_1, m_1, n_1) and (l_2, m_2, n_2), is the diametral plane of chords parallel

to (l_3, m_3, n_3). Accordingly a set of three planes, chosen in this way, has the property that each plane bisects all chords drawn parallel to the line of intersection of the other two. It is called a set of conjugate planes of the surface, and the set of three lines may be called a set of conjugate lines. For an ellipsoid these lines are called conjugate diameters; and they may also be called conjugate diameters in the case of a hyperboloid, if it is understood that this does not imply that they all meet the surface. In the case of a cone these lines, drawn from the vertex, do not meet the surface again.

The construction shows that, for a given surface, there are any number of sets of conjugate planes and lines; and that any given plane or line, through the centre, is a member of any number of sets. Principal planes and axes are sets of conjugate planes and lines at right angles.

It will be seen that a hyperboloid,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

and its conjugate,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = -1,$$

have the same sets of conjugate planes and lines. And if we regard this pair of surfaces as a single surface, which may be a useful thing to do, every member of a set of conjugate lines meets this surface, and is a diameter of definite length.

135. *Tangent Plane.* If the point P, (x', y', z'), is on the surface

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

the directions of the lines, if any, through this point, which lie on the surface, and are therefore reckoned as tangent lines of the surface, are given by the equations

$$K = 0, \quad \alpha x'l + \beta y'm + \gamma z'n = 0.$$

And the other tangent lines at P are the tangents of sections of the surface by planes through P.

If (l, m, n) are the direction cosines of a chord, PQ, of one of these sections, the length of this chord is the positive value of

$$\pm \frac{2}{K}(\alpha x'l + \beta y'm + \gamma z'n).$$

Therefore the limiting position of this chord, when Q tends to coincidence with P, is one in which

$$\alpha x'l + \beta y'm + \gamma z'n = 0.$$

Thus all the tangent lines at P are the lines, through P, in a plane at right angles to the direction $(\alpha x', \beta y', \gamma z')$. This is the tangent plane at P, and its equation is

$$\alpha x'(x - x') + \beta y'(y - y') + \gamma z'(z - z') = 0$$

or

$$\alpha x'x + \beta y'y + \gamma z'z = 1.$$

Therefore the equations of the normal, at the same point, are

$$x = x' + \alpha x'\rho, \quad y = y' + \beta y'\rho, \quad z = z' + \gamma z'\rho,$$

where ρ is a parameter.

If p is the length, and (λ, μ, ν) are the direction cosines, of the perpendicular, from the centre, on the tangent plane at a point (x', y', z'), the equation of the plane may be written

$$\lambda x + \mu y + \nu z = p.$$

Therefore $\quad \lambda = p\alpha x', \quad \mu = p\beta y', \quad \nu = p\gamma z'.$

Therefore $$\frac{1}{p^2} = \alpha^2 x'^2 + \beta^2 y'^2 + \gamma^2 z'^2,$$

and $$p^2 = \frac{\lambda^2}{\alpha} + \frac{\mu^2}{\beta} + \frac{\nu^2}{\gamma}.$$

136. *Polar Plane.* If the point P, (x', y', z'), is not on the surface, draw through P a straight line,

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

cutting the surface in two points; and let ρ_1, ρ_2 be the

values of ρ for these two points. And let Q, (x'', y'', z''), be the point on this line given by

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2},$$

as in § 122. Then the quadratic equation,

$$K\rho^2 + 2(\alpha x'l + \beta y'm + \gamma z'n)\rho + D = 0,$$

the roots of which are ρ_1, ρ_2, shows that this is equivalent to

$$\rho = -\frac{D}{\alpha x'l + \beta y'm + \gamma z'n}.$$

Also $\quad l\rho = x'' - x', \quad m\rho = y'' - y', \quad n\rho = z'' - z';$

therefore

$$\alpha x'(x'' - x') + \beta y'(y'' - y') + \gamma z'(z'' - z') = -D.$$

This shows that all the points, Q, chosen in this way, for different directions of the line through P, are on the plane

$$\alpha x'(x - x') + \beta y'(y - y') + \gamma z'(z - z') + D = 0.$$

But $\quad D = \alpha x'^2 + \beta y'^2 + \gamma z'^2 - 1,$

therefore the equation is

$$\alpha x'x + \beta y'y + \gamma z'z = 1.$$

The plane represented by this equation is called the polar plane of the point (x', y', z') with regard to the surface, and this point is called the pole of the plane.

Thus the polar plane of a point on the surface is the tangent plane at that point.

137 *Polar Lines.* The definition of polar lines is the same for any quadric as for a sphere. That is to say, PQ and RS are polar lines if each of them is the line of intersection of the polar planes of all points on the other. Take any straight line, PQ, whose equations are

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho.$$

Each value of ρ specifies a point on the line, whose polar plane with regard to the quadric $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is

$$\alpha (x' + l\rho) x + \beta (y' + m\rho) y + \gamma (z' + n\rho) z = 1,$$

or $$\alpha x'x + \beta y'y + \gamma z'z - 1 + \rho (\alpha lx + \beta my + \gamma nz) = 0.$$

Thus the polar plane of each point on PQ passes through the line of intersection of the two planes

$$\alpha x'x + \beta y'y + \gamma z'z - 1 = 0,$$
$$\alpha lx + \beta my + \gamma nz = 0.$$

Accordingly this pair of equations represents RS, the polar line of PQ.

138. *Cone.* For the cone $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$ the procedure of § 136 gives $\alpha x'x + \beta y'y + \gamma z'z = 0$ as the equation of the polar plane of the point (x', y', z').

By comparing this equation with the relation between conjugate lines, (§ 134), it will be seen that the relation between a set of conjugate lines, OP, OQ, OR, is that the plane through any two of them is the polar plane of every point on the third. Hence we have another useful form of statement of the same thing, namely that, if P, Q and R are in one plane, each of these points is the pole of the line joining the other two with regard to the conic which is the section of the cone by this plane. This follows from the fact that, in plane geometry, the polar line of a point with regard to a conic is given by the equation

$$\rho = 2\rho_1\rho_2/(\rho_1 + \rho_2).$$

139. *Rectilinear Generators.* The condition that straight lines can be drawn on the surface

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

can be found from the equations

$$\alpha l^2 + \beta m^2 + \gamma n^2 = 0,$$
$$\alpha x'l + \beta y'm + \gamma z'n = 0,$$

which give the direction cosines of the lines on the surface, if any, that can be drawn through a point, (x', y', z').

Let x' be a coordinate which is not zero. By eliminating l, we get a quadratic equation for the ratio m/n, namely

$$(\beta y'm + \gamma z'n)^2 = \alpha x'^2 (\beta m^2 + \gamma n^2) = 0,$$

or $$(\alpha x'^2 + \beta y'^2)\,\beta m^2 + 2\beta\gamma y'z'mn + (\gamma z'^2 + \alpha x'^2)\,\gamma n^2 = 0.$$

This has real roots if

$$\beta^2\gamma^2 y'^2 z'^2 - \beta\gamma(\alpha x'^2 + \beta y'^2)(\gamma z'^2 + \alpha x'^2)$$

is positive or zero; that is to say, if $-\alpha\beta\gamma x'^2$ is positive or zero. And the solution of the quadratic equation is

$$\beta(\alpha x'^2 + \beta y'^2)\frac{m}{n} = -\beta\gamma y'z' \pm x'\sqrt{(-\alpha\beta\gamma)};$$

whence the equation

$$\alpha x'l + \beta y'm + \gamma z'n = 0$$

gives the corresponding value of the ratio l/n.

This shows that there are real straight lines on the surface if, and not unless, $\alpha\beta\gamma$ is negative. At any point of any hyperboloid the cone K exists; but the tangent plane cuts it in two straight lines only if $\alpha\beta\gamma$ is negative, that is to say if it is a hyperboloid of one sheet.

If $\alpha\beta\gamma$ were zero, a case which has been excluded, the surface would be a cylinder.

As we are concerned only with a hyperboloid of one sheet, let us write its equation, as in § 89,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad (a, b \text{ and } c \text{ positive}),$$

or $$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

This form of the equation shows that all points on the line of intersection of the planes

$$\frac{x}{a} + \frac{z}{c} = \lambda\left(1 + \frac{y}{b}\right) \quad \text{and} \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\lambda}\left(1 - \frac{y}{b}\right)$$

are on the surface. Thus by giving different values to λ we get a system of straight lines on the surface. Similarly we have another system of straight lines on the

surface, namely those obtained by giving different values to μ in the equations

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right).$$

These equations show that for each point on the surface there is a single pair of values for λ and μ, giving two lines through that point; and that every pair of values assigned to λ and μ specifies a single point on the surface, namely the point whose coordinates, (found by solving these equations), are

$$\frac{\lambda+\mu}{1+\lambda\mu}a,\quad \frac{1-\lambda\mu}{1+\lambda\mu}b,\quad \frac{\lambda-\mu}{1+\lambda\mu}c.$$

Thus each line of one system intersects every line of the other system, except one which is parallel to it, the point of intersection passing to infinity. And every line intersects the plane of xy at a point for which $\lambda=\mu$. Each system of lines is a set of generating lines, or rectilinear generators, of the surface. See §§ 87–89.

The equations of the lines on the surface can be found in the form given in § 89, as follows. The equations for the direction cosines of the lines drawn through a point (x', y', z') are

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0,\quad \frac{x'l}{a^2}+\frac{y'm}{b^2}-\frac{z'n}{c^2}=0.$$

Every line meets the plane of xy, for n cannot be zero. Take $(a\cos\phi,\ b\sin\phi,\ 0)$ as the coordinates of a point on the section of the surface by this plane. For a line through this point

$$\frac{l\cos\phi}{a}+\frac{m\sin\phi}{b}=0.$$

Thus $l=ka\sin\phi$, and $m=-kb\cos\phi$, and the first equation gives $n=\pm kc$. Thus the equations of the two lines through the point in question are

$$\frac{x-a\cos\phi}{a\sin\phi}=\frac{y-b\sin\phi}{-b\cos\phi}=\pm\frac{z}{c}.$$

Each choice of a sign gives one of the two lines.

140. *Circular Sections.* The quadric,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

is a surface of revolution if two of the coefficients, α, β, γ, are equal; and its circular plane sections are then the sections by planes at right angles to the axis of revolution.

To find the circular sections of this surface by planes through the centre, when α, β, γ are all unequal, draw the sphere

$$\lambda (x^2 + y^2 + z^2) = 1,$$

where λ is some positive number, also the cone

$$\alpha x^2 + \beta y^2 + \gamma z^2 - \lambda (x^2 + y^2 + z^2) = 0.$$

This cone, with vertex at the centre, cuts the sphere at the points of its intersection with the quadric, and does not exist if the sphere and the quadric do not meet. And the quadric has central circular plane sections if, and not unless, λ can be chosen so that the cone is a pair of planes. Now the condition that the cone is a pair of planes is that one of the three coefficients, $\alpha - \lambda$, $\beta - \lambda$, $\gamma - \lambda$, is zero, while the other two have opposite signs. That is to say, if the numbers α, β, γ are arranged in order of magnitude, λ must be equal to the middle one, which must therefore be positive.

Thus for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ with } a > b > c,$$

we have, as explained in § 77, the pair of planes

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right) x^2 - \left(\frac{1}{c^2} - \frac{1}{b^2}\right) z^2 = 0,$$

giving central circular sections, and the other circular sections are the sections by planes parallel to these.

For the hyperboloid of one sheet,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \text{ with } a > b,$$

the value to be assigned to λ is $\frac{1}{a^2}$, and we get circular sections by the pair of planes

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2-\left(\frac{1}{c^2}+\frac{1}{a^2}\right)z^2=0,$$

and by all planes parallel to them. These two surfaces cannot have any other circular plane sections, because all planes through the centre intersect them.

For the hyperboloid of two sheets,

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

there is no value of λ which makes the cone a pair of planes; and this agrees with the obvious fact that there are no central circular sections. The circular sections of this surface are its sections by planes which cut the conjugate surface,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

in circles, (§ 132). Similarly the circular sections of the asymptotic cone are given by the same series of planes.

141. The sphere, $\lambda(x^2+y^2+z^2)=1$, is said to touch the quadric, $\alpha x^2+\beta y^2+\gamma z^2=1$ at any point at which they have a common tangent plane. Assuming, as before, that α, β and γ are unequal, this occurs in two ways, namely in the limiting case of the cone

$$\alpha x^2+\beta y^2+\gamma z^2-\lambda(x^2+y^2+z^2)=0$$

being a single straight line, and in the case in which this cone is a pair of intersecting planes, the points of contact then being in the line of intersection of the planes. Thus the sphere touches the quadric when λ, which must be positive, is equal to α or β or γ. Thus we have three cases for an ellipsoid, two for a hyperboloid of one sheet, and one for a hyperboloid of two sheets, as is obvious from the forms of these surfaces.

142. *Axes of a Central Section.* The case of an ellipsoid has already been dealt with, (§ 78). In dealing with sections of the general central quadric,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

which may be hyperbolas having conjugate axes which do not meet the surface, let us adopt a different procedure, namely that of finding maximum and minimum values of

$$\alpha l^2 + \beta m^2 + \gamma n^2,$$

which will be denoted by $1/R$, where l, m, n are the direction cosines of a radius, drawn from the centre, in a given plane

$$\lambda x + \mu y + \nu z = 0.$$

Thus l, m, n are subject to the conditions

$$\lambda l + \mu m + \nu n = 0,$$
$$l^2 + m^2 + n^2 = 1.$$

If R is positive it is the square of a radius of the given surface; and if R is negative, $-R$ is the square of a radius of the conjugate surface. In each case (l, m, n) are the direction cosines of the radius in question.

By differentiation we get the conditions that R is a maximum or minimum, namely

$$\begin{aligned} \alpha l dl + \beta m dm + \gamma n dn &= 0,\\ \lambda dl + \mu dm + \nu dn &= 0,\\ l dl + m dm + n dn &= 0. \end{aligned}$$

Following the usual procedure of differential calculus, we multiply the second and third equations by k and k', and add; then k and k' can be found such that

$$\begin{aligned} \alpha l + k\lambda + k'l &= 0,\\ \beta m + k\mu + k'm &= 0,\\ \gamma n + k\nu + k'n &= 0; \end{aligned}$$

and multiplying by l, m and n, and adding, we get

$$\frac{1}{R} + k' = 0.$$

Therefore
$$\left(\alpha - \frac{1}{R}\right) l + k\lambda = 0,$$
$$\left(\beta - \frac{1}{R}\right) m + k\mu = 0,$$
$$\left(\gamma - \frac{1}{R}\right) n + k\nu = 0.$$

Substituting the values of $l : m : n$ thus obtained in the equation $\lambda l + \mu m + \nu n = 0$, we get
$$\frac{\lambda^2}{\alpha R - 1} + \frac{\mu^2}{\beta R - 1} + \frac{\nu^2}{\gamma R - 1} = 0.$$

This is a quadratic equation whose roots, R_1, R_2, are the maximum and minimum values of R. And the corresponding values of the ratios $l : m : n$ are
$$\frac{\lambda}{\alpha R_1 - 1} : \frac{\mu}{\beta R_1 - 1} : \frac{\nu}{\gamma R_1 - 1} \quad \text{and} \quad \frac{\lambda}{\alpha R_2 - 1} : \frac{\mu}{\beta R_2 - 1} : \frac{\nu}{\gamma R_2 - 1}.$$

The quadratic equation gives
$$R_1 R_2 = \frac{1}{\alpha\beta\gamma}\left(\frac{\lambda^2}{\alpha} + \frac{\mu^2}{\beta} + \frac{\nu^2}{\gamma}\right)^{-1}.$$

Let us write P for $\dfrac{\lambda^2}{\alpha} + \dfrac{\mu^2}{\beta} + \dfrac{\nu^2}{\gamma}$; then $R_1 R_2 P = \dfrac{1}{\alpha\beta\gamma}$. If P is positive it is the square of the perpendicular from the centre on the tangent plane parallel to the plane of the section. If P is negative, $-P$ is the square of the perpendicular from the centre on the tangent plane of the conjugate surface, parallel to the plane of the section.

143. *Paraboloids.* Examination of the general equation of the second order, in the previous chapter, has shown that the only quadric surfaces which have no centre are the paraboloids and the parabolic cylinder.

The equations of the paraboloids, referred to the two principal planes, (§ 124), and the tangent plane at the vertex, as coordinate planes, are, as in §§ 89 and 91,
$$\frac{x^2}{L} + \frac{y^2}{L'} = 2z \quad \text{and} \quad \frac{x^2}{L} - \frac{y^2}{L'} = 2z,$$

where L and L' are positive numbers. These equations represent, respectively, the elliptic and the hyperbolic paraboloid; and $2L$ and $2L'$ are the latera recta of their sections by the principal planes. It has been shown, (§ 91), that the hyperbolic paraboloid has rectilinear generators; and it is obvious from the construction of this surface that it has no circular sections.

The planes of circular sections of the elliptic paraboloid are parallel to those of the cylinder

$$\frac{x^2}{L} + \frac{y^2}{L'} = 1,$$

because the equations of these surfaces have the same terms of the second order, (§ 132). Let L be the greater of the two numbers L and L', if they are not equal. Now all the sections of the cylinder, by planes inclined to the axis of z, are ellipses or circles; and the lengths of the semi-axes of the section by the plane of xy are $\sqrt{L}$ and $\sqrt{L'}$. Therefore the planes of circular sections of the cylinder are parallel to the planes $z = \pm y \tan\theta$, where θ is given by the equation $\sqrt{L'} = \sqrt{L}\cos\theta$. Therefore the planes of the circular sections of the elliptic paraboloid are parallel to the planes

$$z = \pm y \sqrt{\left(\frac{L - L'}{L'}\right)}.$$

The tangent and polar planes for the paraboloid,

$$ax^2 + by^2 = 2z,$$

can be found in the same way as for other surfaces. The equation which gives the values of ρ for the points in which the straight line

$$x = x' + l\rho, \quad y = y' + m\rho, \quad z = z' + n\rho,$$

meets the surface, is

$$(al^2 + bm^2)\,\rho^2 + 2\,(ax'l + by'm - n)\,\rho + ax'^2 + by'^2 - 2z' = 0.$$

This shows, by the same procedure as in § 135, that at

any point (x', y', z'), on the surface, there is a tangent plane represented by the equation

$$ax'(x - x') + by'(y - y') - (z - z') = 0,$$

which may be written

$$ax'x + by'y = z + z';$$

also that the latter equation, when the point (x', y', z') is not on the surface, represents the polar plane of this point, specified, in the same way as before, by the equation

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}.$$

The rectilinear generators of the hyperbolic paraboloid,

$$\frac{x^2}{L} - \frac{y^2}{L'} = 2z;$$

can be found by writing this equation in the form

$$\left(\frac{x}{\sqrt{L}} + \frac{y}{\sqrt{L'}}\right)\left(\frac{x}{\sqrt{L}} - \frac{y}{\sqrt{L'}}\right) = 2z.$$

This equation shows that, for any value of λ, the line of intersection of the planes

$$\frac{x}{\sqrt{L}} + \frac{y}{\sqrt{L'}} = \lambda, \quad \frac{x}{\sqrt{L}} - \frac{y}{\sqrt{L'}} = \frac{2z}{\lambda},$$

is on the surface. Also that, for any value of μ, the line of intersection of the planes

$$\frac{x}{\sqrt{L}} - \frac{y}{\sqrt{L'}} = \mu, \quad \frac{x}{\sqrt{L}} + \frac{y}{\sqrt{L'}} = \frac{2z}{\mu},$$

is on the surface. Thus there are two systems of lines on the surface. Every value of λ gives a line of one system, and every value of μ gives a line of the other system. And any pair of values of λ and μ specifies a point on the surface, and the line of each system which passes through that point. Also the equations,

$$\frac{x}{\sqrt{L}} + \frac{y}{\sqrt{L'}} = \lambda, \quad \frac{x}{\sqrt{L}} - \frac{y}{\sqrt{L'}} = \mu,$$

show that all lines of the first system are parallel to a certain plane, and that all lines of the second system are parallel to another plane. See § 94, 2.

144. *Confocal Quadrics.* If a^2, b^2 and c^2 are given positive numbers, the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

represents a system of quadrics which are called confocal quadrics, distinguished from one another by the value of λ, which is called the parameter of the system.

Let us take $a^2 > b^2 > c^2$. Then values of λ greater than $-c^2$ give ellipsoids, values between $-c^2$ and $-b^2$ give hyperboloids of one sheet, and values between $-b^2$ and $-a^2$ give hyperboloids of two sheets.

The values of λ for surfaces of this system which pass through a given point, (f, g, h), if f, g and h are none of them zero, are given by the equation

$$f^2(b^2+\lambda)(c^2+\lambda)+g^2(c^2+\lambda)(a^2+\lambda)+h^2(a^2+\lambda)(b^2+\lambda)$$
$$-(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)=0.$$

If we substitute for λ, in succession, the values $-a^2$, $-b^2$, $-c^2$, ∞, the signs of the successive values of the left-hand side of this equation are respectively $+, -, +, -$. Therefore this equation for λ has three real roots, one between $-a^2$ and $-b^2$, one between $-b^2$ and $-c^2$, and one greater than $-c^2$. Therefore three quadrics of the system, and no more, pass through the given point, one an ellipsoid, one a hyperboloid of one sheet, and one a hyperboloid of two sheets. Thus a point in a given octant may be specified by the parameters, say $\lambda_1, \lambda_2, \lambda_3$, of the confocals that pass through it.

Any two intersecting quadrics of the system intersect at right angles. To prove this, let the equations of the quadrics be

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1 \quad \text{and} \quad \frac{x^2}{A+\mu}+\frac{y^2}{B+\mu}+\frac{z^2}{C+\mu}=1,$$

and let (f, g, h) be a point in which they intersect. Then

$$\frac{f^2}{A}+\frac{g^2}{B}+\frac{h^2}{C}=\frac{f^2}{A+\mu}+\frac{g^2}{B+\mu}+\frac{h^2}{C+\mu},$$

or $$\frac{f^2}{A(A+\mu)}+\frac{g^2}{B(B+\mu)}+\frac{h^2}{C(C+\mu)}=0.$$

That is to say, the direction $\left(\frac{f}{A}, \frac{g}{B}, \frac{h}{C}\right)$ is at right angles to the direction $\left(\frac{f}{A+\mu}, \frac{g}{B+\mu}, \frac{h}{C+\mu}\right)$. But these are the directions of the normals of the two surfaces at the point (f, g, h). This shows that the three surfaces of the system which can be drawn through any given point are at right angles to one another.

145. Let (x', y', z') be a given point on a given quadric
$$\alpha x^2+\beta y^2+\gamma z^2=1.$$
The plane conjugate to the diameter through this point is
$$\alpha x'x+\beta y'y+\gamma z'z=0.$$
And R_1, R_2, (§ 142), specifying the squares of the semi-axes of the section of the surface by this plane, are the roots of the quadratic equation
$$\frac{\alpha^2 x'^2}{\alpha R-1}+\frac{\beta^2 y'^2}{\beta R-1}+\frac{\gamma^2 z'^2}{\gamma R-1}=0.$$
Let us now find the confocal quadrics through the same point. Their equations are
$$\frac{x^2}{\frac{1}{\alpha}-\lambda_1}+\frac{y^2}{\frac{1}{\beta}-\lambda_1}+\frac{z^2}{\frac{1}{\gamma}-\lambda_1}=1,$$
$$\frac{x^2}{\frac{1}{\alpha}-\lambda_2}+\frac{y^2}{\frac{1}{\beta}-\lambda_2}+\frac{z^2}{\frac{1}{\gamma}-\lambda_2}=1,$$
where λ_1, λ_2 are the values of λ given by the equation
$$\frac{x'^2}{\frac{1}{\alpha}-\lambda}+\frac{y'^2}{\frac{1}{\beta}-\lambda}+\frac{z'^2}{\frac{1}{\gamma}-\lambda}=\alpha x'^2+\beta y'^2+\gamma z'^2,$$
which may be written
$$\frac{\alpha^2 x'^2}{\alpha\lambda-1}+\frac{\beta^2 y'^2}{\beta\lambda-1}+\frac{\gamma^2 z'^2}{\gamma\lambda-1}=0.$$
Therefore λ_1, λ_2 are the same as R_1, R_2.

Also the direction cosines, l, m, n, of the semi-axis represented by R_1, are given, (§ 142), by the equations

$$\frac{l}{x'}\left(\frac{1}{\alpha} - R_1\right) = \frac{m}{y'}\left(\frac{1}{\beta} - R_1\right) = \frac{n}{z'}\left(\frac{1}{\gamma} - R_1\right).$$

Therefore this axis is parallel to the normal at the point (x', y', z') of the confocal specified by R_1. Similarly the semi-axis represented by R_2 is parallel to the normal at this point of the confocal specified by R_2.

146. *Notes and Examples.*

1. The equation, (§ 72), of a cone enveloping the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

is $$(x'^2/a^2 + y'^2/b^2 + z'^2/c^2 - 1)(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1) - (x'x/a^2 + y'y/b^2 + z'z/c^2 - 1)^2 = 0,$$

where (x', y', z') are the coordinates of the vertex, A. It will now be proved that the principal axes of this cone are the normals at A of the three confocal quadrics through this point, and that its equation referred to these axes is $$x^2/\lambda_1 + y^2/\lambda_2 + z^2/\lambda_3 = 0,$$

where λ_1, λ_2, λ_3 are the parameters of these confocals.

The plane of contact of the cone is the polar plane of A with regard to the ellipsoid. Let us call this the plane B. Draw the normals AP, AQ, AR of the three confocals, meeting the plane B in P, Q, R. Then these three lines are at right angles. Now the equation of the plane AQR, the tangent plane of the confocal λ_1, is

$$x'x/(a^2 + \lambda_1) + y'y/(b^2 + \lambda_1) + z'z/(c^2 + \lambda_1) = 1;$$

and the coordinates of the pole of this plane with regard to the ellipsoid are $a^2x'/(a^2 + \lambda_1)$, $b^2y'/(b^2 + \lambda_1)$, $c^2z'/(c^2 + \lambda_1)$, which obviously satisfy the equations of the line AP, namely

$$(x - x')\frac{a^2 + \lambda_1}{x'} = (y - y')\frac{b^2 + \lambda_1}{y'} = (z - z')\frac{c^2 + \lambda_1}{z'}.$$

Therefore the pole of the plane AQR is on the line AP. And it must also be in the plane B; therefore it is the point P. Similarly Q and R are the poles of the planes ARP and APQ respectively. Therefore, in the plane B, each of the points P, Q, R is the pole, with regard to the section of the ellipsoid by this plane, of the line joining the other two.

But the section of the ellipsoid by the plane B is also a section of the cone. Therefore, in accordance with § 138, AP, AQ, AR are a set of conjugate lines of the cone. And they are at right angles, therefore they are the principal axes.

The coefficients in the equation of the cone referred to these axes can be found in various ways. If the discriminating cubic is employed, a comparison of this equation with the cubic equation for λ in § 144, so as to show that the reciprocals of its roots are proportional to λ_1, λ_2, λ_3, involves no difficulty.

2. To find the lengths of the axes of the section of the quadric

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

or $\phi(x, y, z) = 1$, by a given plane, $\lambda x + \mu y + \nu z = 0$, the procedure of § 142 can be followed exactly, with the result that the quadratic equation whose roots are R_1 and R_2 is

$$\Phi(\lambda, \mu, \nu) R^2 - \{(a + b + c)(\lambda^2 + \mu^2 + \nu^2) - \phi(\lambda, \mu, \nu)\} R + \lambda^2 + \mu^2 + \nu^2 = 0,$$

where Φ has the same meaning as in § 111. Also the direction cosines (l, m, n) of an axis are given by the equations

$$\frac{1}{\lambda}\{(a - 1/R) l + hm + gn\} = \frac{1}{\mu}\{hl + (b - 1/R) m + fn\}$$
$$= \frac{1}{\nu}\{gl + fm + (c - 1/R) n\}.$$

3. To find the central circular sections of the quadric $\phi(x, y, z) = 1$, the procedure of § 140 is followed. Let α, β, γ be the roots of the discriminating cubic, and $\alpha < \beta < \gamma$. Then the sphere $x^2 + y^2 + z^2 = 1/\beta$ cuts the quadric in the required sections, which exist if, and not unless β is positive. And the equation of the pair of planes is

$$\phi(x, y, z) - \beta(x^2 + y^2 + z^2) = 0.$$

4. Normals are drawn to a hyperboloid of one sheet at points lying along a given generating line, show that these normals are all parallel to a certain plane.

5. The normal at a variable point, P, of a central quadric cuts a principal section in G. Prove that the locus of a point which divides PG in a constant ratio is a quadric. (C.)

6. Prove that the condition that the section of the surface

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = a^2$$

by the plane $lx + my + nz = p$ is a parabola is

$$mn + nl + lm = 0.$$

[Elimination of z gives the projection of the section on the plane of xy, and this must be a parabola.]

7. Prove that, if $x^2/(b + c) + y^2/(c + a) + z^2/(a + b) = 0$ represents a cone, its tangent planes cut the surface $ax^2 + by^2 + cz^2 = 1$ in rectangular hyperbolas.

8. Show that the right circular cylinders described about the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \quad (a > b > c)$$

are represented by the equation

$$(b^2 - c^2)\,x^2 - (c^2 - a^2)\,y^2 + (a^2 - b^2)\,z^2 \pm 2\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}\,zx = (a^2 - c^2)\,b^2.$$

9. Calculate the lengths of the principal axes of the section of the quadric $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 20$ by the plane

$$x + y + z = 1.$$

10. Prove that, if the quadric

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

is a circular cylinder,

$$4f^2 = a^2 - (b - c)^2, \quad 4g^2 = b^2 - (c - a)^2, \quad 4h^2 = c^2 - (a - b)^2;$$

and the radius of a cross section is given by $2/r^2 = a + b + c$. (C.)

11. Prove that a plane which cuts the hyperboloid

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1, \quad (a > b),$$

in a circle of radius r, cuts the asymptotic cone in a circle of radius $\sqrt{(r^2 - a^2)}$. (C.)

12. Prove that, if ψ is the angle between the generating lines of the hyperboloid, $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, which pass through a point at distance r from the centre, and p is the perpendicular from the centre on the plane containing these lines,

$$2abc \cot \psi = p\,(r^2 - a^2 - b^2 - c^2).$$

13. A system of confocal quadrics being given, also any plane; show that the locus of the poles of this plane with regard to the quadrics is a straight line at right angles to the plane; and that this line is at right angles to its polar line with regard to any quadric of the system. (C.)

14. Find the generating lines of the surface $mzx = cy$, and transform the equation to the principal axes. (C.)

15. Take any two finite straight lines, not in one plane, and divide them into the same number of equal parts. Show that lines, or threads, joining the corresponding points of division, are generators of a portion of a hyperbolic paraboloid, and thus form a model of a portion of this surface.

CHAPTER XI

TWISTED CURVES AND DEVELOPABLES

147. A curve is generated by the motion of a point with reference to a set of coordinate axes, or some other given base. If the curve is not in one plane it is said to have torsion, and may be called a twisted curve.

Let a point, P, on a given curve, be specified by its distance, s, measured along the curve, in a given direction, from a given point on it; and let (l, m, n) be the direction cosines of the tangent at P, reckoned in the same direction along the curve. It is assumed here that we are dealing with a curve at each point of which there is a tangent, and which is such that l, m and n are functions of s, possessing differential coefficients of the first and higher orders, so far as we have occasion to use them. But s may be replaced, as independent variable, by one of the coordinates of P, or by some parameter, say θ, of which the coordinates, x, y, z, of a point on the curve are given functions.

The differential calculus supplies the rules, (§ 163), for a change of the independent variable. Accordingly the choice of independent variable for the purpose of any particular investigation is only a matter of convenience.

A curve might be conceived as the path, with reference to the earth, of the centre of gravity of an aeroplane, starting from a given point, and having its speed and direction of steering recorded at every instant. And time might then be introduced as independent variable.

148. *Tangent.* The tangent of a given curve, at a point P, is the limiting case of a straight line drawn through P and an adjacent point Q on the curve, when Q tends to coincidence with P. If (x, y, z) are the coordinates of the point P, specified by s, and $(x + \delta x,\ y + \delta y,\ z + \delta z)$ the coordinates of Q, specified by $s + \delta s$, the direction cosines of PQ are proportional to, and have the same signs as,

δx, δy, δz. Therefore, by the same procedure as in plane geometry, l, m, n are found to be equal to $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$.

149. *Osculating plane.* The general equation of a plane drawn through a given point P, (a, b, c), on a given curve, is $A(x-a) + B(y-b) + C(z-c) = 0.$

The condition that the plane is drawn through the tangent at P, direction cosines (l, m, n), is

$$Al + Bm + Cn = 0.$$

And the plane can be chosen so as to satisfy also another condition. Let us make it parallel to another tangent of the curve, at a point Q, with direction cosines $(l + \delta l, m + \delta m, n + \delta n)$. The condition for this is

$$A(l + \delta l) + B(m + \delta m) + C(n + \delta n) = 0,$$

that is to say $\quad A\delta l + B\delta m + C\delta n = 0.$

We then have two equations which give the ratios $A : B : C$, and the plane is uniquely specified.

If P is the point specified by s, and Q the point specified by $s + \delta s$; the second condition may be written

$$A\frac{\delta l}{\delta s} + B\frac{\delta m}{\delta s} + C\frac{\delta n}{\delta s} = 0;$$

and in the limiting case in which Q tends to coincidence with P, this equation is

$$A\frac{dl}{ds} + B\frac{dm}{ds} + C\frac{dn}{ds} = 0;$$

thus the ratios $A : B : C$ are the values at the point (a, b, c) of the ratios

$$m\frac{dn}{ds} - n\frac{dm}{ds} \ : \ n\frac{dl}{ds} - l\frac{dn}{ds} \ : \ l\frac{dm}{ds} - m\frac{dl}{ds}.$$

The plane thus specified is called the osculating plane of the curve at P, because it is the plane which has the closest contact with the curve at that point. In the case of a plane curve it is the plane of the curve. In the case of a twisted curve the osculating plane suffers angular

displacement as we pass along the curve, by turning about the tangent. This is the thing that constitutes the torsion of the curve. The angular displacement of the tangent, as we pass along the curve, is the thing that constitutes its curvature.

The tangents of the curve generate a developable surface, of which the osculating planes are obviously tangent planes. The curve is called the edge of regression of this surface. The surface is a developable, (§ 22), because its tangent planes have only one degree of freedom. A model of the surface, made of flexible material, could be developed into a plane by bending about the generating lines; the curve being thus converted into a plane curve.

150. *Normals*. The plane through a point, P, on the curve, at right angles to the tangent at that point, is called the normal plane of the curve at P. Every straight line drawn through P in the normal plane is at right angles to the tangent, and is therefore called a normal line of the curve at P. The normal line in the osculating plane is called the principal normal. The normal line at right angles to the osculating plane is called the binormal.

The diagram shows the tangent, PT, drawn in the direction in which s is measured, with direction cosines (l, m, n); and the principal normal, PN, drawn in the direction of the concavity of the curve, with direction cosines (l', m', n'); and the binormal, PB, with direction cosines (λ, μ, ν), drawn so that the directions PT, PN, PB, in this order, are conformable, (§ 8), with the directions of the axes of x, y and z. These nine cosines, connecting two sets of lines at right angles, have all the numerous relations given in § 50; for example

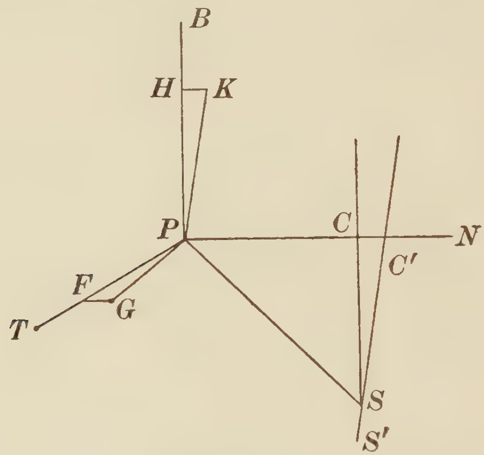
$$l^2 + m^2 + n^2 = 1, \ldots\ldots,$$

$$l^2 + l'^2 + \lambda^2 = 1, \ldots\ldots,$$

$$l'\lambda + m'\mu + n'\nu = 0, \ldots\ldots,$$

$$l = m'\nu - \mu n', \quad l' = \mu n - m\nu, \quad \lambda = mn' - m'n, \quad \ldots\ldots$$

151. *Curvature.* The point P being specified by s, take an adjacent point, Q, on the curve, specified by $s + \delta s$, where δs is positive, and let $\delta\psi$ be the positive acute angle between the tangents at P and Q. Then the value of $\frac{\delta\psi}{\delta s}$ in the limit, when Q tends to coincidence with P, namely $\frac{d\psi}{ds}$, is called the curvature at P, and will be denoted by $\frac{1}{\rho}$. And ρ is called the radius of curvature. Thus ρ is for the present defined as positive. Taking δs and $\delta\psi$ both negative would give the same result.



On PT take a point F at unit distance from P. And draw PG, of unit length, parallel to the tangent at the adjacent point Q, specified by $s + \delta s$, at which the tangent, principal normal and binormal have direction cosines

$$l + \delta l, \quad m + \delta m, \quad n + \delta n,$$
$$l' + \delta l', \quad m' + \delta m', \quad n' + \delta n',$$
$$\lambda + \delta\lambda, \quad \mu + \delta\mu, \quad \nu + \delta\nu.$$

Then the projections of PF on the axes are l, m, n; and the projections of PG are $l + \delta l$, $m + \delta m$, $n + \delta n$. Thus the direction cosines of the line FG are proportional to, and have the same signs as, δl, δm, δn. And in the limit,

when Q tends to coincidence with P, FG is the same direction as PN. Also, (§ 7),

$$dl^2 + dm^2 + dn^2 = d\psi^2 = \frac{1}{\rho^2} ds^2,$$

therefore, as ρ is positive,

$$l' = \rho \frac{dl}{ds}, \quad m' = \rho \frac{dm}{ds}, \quad n' = \rho \frac{dn}{ds}.$$

This may be written

$$l' = \rho \frac{d^2x}{ds^2}, \quad m' = \rho \frac{d^2y}{ds^2}, \quad n' = \rho \frac{d^2z}{ds^2}.$$

Therefore $$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

152. *Torsion.* Let $\delta\epsilon$ be the acute angle between the binormals at P and Q, measured so that the sign of $\frac{\delta\epsilon}{\delta s}$ depends on whether the angular displacement of the osculating plane, as we pass, in either direction, along the curve, makes a right-handed or a left-handed screw with that direction, see § 185. Then $\frac{d\epsilon}{ds}$ is called the torsion of the curve at P, and will be denoted by $\frac{1}{\sigma}$. The convention which will be adopted here, for the present, is that σ is positive when the screw referred to in this definition is left-handed. The torsion of the helix forming an ordinary corkscrew is, according to this convention, negative.

Consideration of the developable surface indicates that the curve is fully specified if the curvature and torsion are given as functions of s; the curvature at each point being the same as the curvature, at the corresponding point, of the plane curve obtained by the development of this surface.

Take a point H on the binormal at unit distance from P, and draw PK of unit length in the direction of the binormal at Q. Then the projections of PH on the axes are λ, μ, ν; and the projections of PK are $\lambda + \delta\lambda$, $\mu + \delta\mu$, $\nu + \delta\nu$. Thus the direction cosines of HK are proportional

to, and have the same signs as, $\delta\lambda$, $\delta\mu$, $\delta\nu$. And in the limit, when Q tends to coincidence with P,

$$d\lambda^2 + d\mu^2 + d\nu^2 = d\epsilon^2 = \frac{1}{\sigma^2} ds^2.$$

If σ is positive $\frac{d\epsilon}{ds}$ is positive, and according to our convention HK has the same direction as PN, as is shown in the diagram, therefore

$$l' = \sigma \frac{d\lambda}{ds}, \qquad m' = \sigma \frac{d\mu}{ds}, \qquad n' = \sigma \frac{d\nu}{ds}.$$

If σ is negative we get the same result, because, although $\frac{d\epsilon}{ds}$ is now negative, this is corrected by K being on the opposite side of H, so that PN and HK are now opposite directions.

The convention as to sign of torsion, which is adopted here, depends on the coordinate axes being drawn in the way which is adopted in this book. If they are drawn in the alternative way, (§ 1), the direction of PB is reversed; and in order to keep the formulae unchanged, the torsion of a curve must be reckoned positive when the screw referred to in the definition is right handed, so that the torsion of the helix forming a corkscrew would be positive. It is convenient to keep the standard formulae unchanged, and to make the convention depend on the way the coordinate axes are drawn.

The notation κ for the curvature, $1/\rho$, and τ for the torsion, $1/\sigma$, is often used.

153. *Frenet's formulae.* Differentiation of the equation

$$l' = \mu n - m\nu, \quad (\text{§ } 150),$$

gives

$$\begin{aligned}\frac{dl'}{ds} &= \mu \frac{dn}{ds} + n \frac{d\mu}{ds} - \nu \frac{dm}{ds} - m \frac{d\nu}{ds} \\ &= \mu \frac{n'}{\rho} + n \frac{m'}{\sigma} - \nu \frac{m'}{\rho} - m \frac{n'}{\sigma} \\ &= -\frac{m'\nu - \mu n'}{\rho} - \frac{mn' - m'n}{\sigma}.\end{aligned}$$

Therefore $$\frac{dl'}{ds} = -\frac{l}{\rho} - \frac{\lambda}{\sigma};$$

and similarly $$\frac{dm'}{ds} = -\frac{m}{\rho} - \frac{\mu}{\sigma},$$

$$\frac{dn'}{ds} = -\frac{n}{\rho} - \frac{\nu}{\sigma}.$$

These three equations, together with the six equations

$$\frac{dl}{ds} = \frac{l'}{\rho}, \quad \frac{dm}{ds} = \frac{m'}{\rho}, \quad \frac{dn}{ds} = \frac{n'}{\rho},$$

$$\frac{d\lambda}{ds} = \frac{l'}{\sigma}, \quad \frac{d\mu}{ds} = \frac{m'}{\sigma}, \quad \frac{d\nu}{ds} = \frac{n'}{\sigma},$$

are called Frenet's formulae.

Now the coordinate axes may have any directions, therefore these formulae show that, if L, M, N are the cosines of the angles which the tangent, principal normal and binormal make with any fixed straight line,

$$\frac{dL}{ds} = \frac{M}{\rho}, \quad \frac{dM}{ds} = -\frac{L}{\rho} - \frac{N}{\sigma}, \quad \frac{dN}{ds} = \frac{M}{\sigma}.$$

154. A symmetrical expression for σ, in terms of the coordinates of a point on the curve, given as functions of s, can be found as follows. We have

$$\lambda = mn' - m'n = \rho\left(m\frac{dn}{ds} - n\frac{dm}{ds}\right),$$

and two similar equations. And differentiation gives the three equations

$$\frac{1}{\rho}\frac{d\lambda}{ds} - \frac{\lambda}{\rho^2}\frac{d\rho}{ds} = m\frac{d^2n}{ds^2} - n\frac{d^2m}{ds^2},$$

$$\frac{1}{\rho}\frac{d\mu}{ds} - \frac{\mu}{\rho^2}\frac{d\rho}{ds} = n\frac{d^2l}{ds^2} - l\frac{d^2n}{ds^2},$$

$$\frac{1}{\rho}\frac{d\nu}{ds} - \frac{\nu}{\rho^2}\frac{d\rho}{ds} = l\frac{d^2m}{ds^2} - m\frac{d^2l}{ds^2}.$$

Also $$l'\lambda + m'\mu + n'\nu = 0,$$

and $$l'\frac{d\lambda}{ds} + m'\frac{d\mu}{ds} + n'\frac{d\nu}{ds} = \frac{1}{\sigma}(l'^2 + m'^2 + n'^2) = \frac{1}{\sigma}.$$

Therefore multiplying the three equations by l', m' and n' respectively, and adding,

$$\frac{1}{\rho\sigma} = l'\left(m\frac{d^2n}{ds^2} - n\frac{d^2m}{ds^2}\right) + m'\left(n\frac{d^2l}{ds^2} - l\frac{d^2n}{ds^2}\right) + n'\left(l\frac{d^2m}{ds^2} - m\frac{d^2l}{ds^2}\right).$$

Therefore

$$\frac{1}{\rho^2\sigma} = \frac{dl}{ds}\left(m\frac{d^2n}{ds^2} - n\frac{d^2m}{ds^2}\right) + \frac{dm}{ds}\left(n\frac{d^2l}{ds^2} - l\frac{d^2n}{ds^2}\right) + \frac{dn}{ds}\left(l\frac{d^2m}{ds^2} - m\frac{d^2l}{ds^2}\right),$$

or

$$-\frac{1}{\rho^2\sigma} = \begin{vmatrix} \dfrac{dx}{ds}, & \dfrac{dy}{ds}, & \dfrac{dz}{ds} \\ \dfrac{d^2x}{ds^2}, & \dfrac{d^2y}{ds^2}, & \dfrac{d^2z}{ds^2} \\ \dfrac{d^3x}{ds^3}, & \dfrac{d^3y}{ds^3}, & \dfrac{d^3z}{ds^3} \end{vmatrix}.$$

155. *Osculating circle and sphere.* Three points on the curve define a circle drawn through them, and four points define a sphere drawn through them. And when these points tend to coincidence at P, the order of them remaining unchanged, the circle becomes a definite circle in the osculating plane, of radius ρ, with its centre on the principal normal; and the sphere assumes a definite position, with its centre in the normal plane. The circle is called the circle of curvature. The sphere is called the osculating sphere, and its radius, R, is called the radius of spherical curvature. In the diagram, C is the centre of the circle of curvature at P, and S is the centre of the sphere. The point C' is the centre of the circle of curvature at an adjacent point Q; and when Q tends to coincidence with P, the line CC' assumes a definite limiting position in the normal plane at P.

These points can be found by drawing normal planes. The normal plane at P is at right angles to PF, and the

normal plane at Q is at right angles to PG; therefore their line of intersection is at right angles to the plane PFG. In the limit, when Q tends to coincidence with P, this line assumes the position CS parallel to the binormal, C being its intersection with the osculating plane, and S the limiting position of the point of intersection of three normal planes.

The diagram represents the case in which ρ increases from P to Q.

156. *Polar Developable.* The normal planes of a curve are the tangent planes of a developable surface, which is called the polar developable of the curve. For a plane curve it is a cylinder. For a twisted curve, its generating lines are the lines CS, $C'S'$, ..., parallel to the binormals, and its edge of regression is the locus of the centre of spherical curvature. The principal normals of this curve are parallel to the principal normals of the given curve. The diagram also gives $d\rho^2 = (CS \, . \, d\epsilon)^2$, and

$$R^2 = PC^2 + CS^2 = \rho^2 + \left(\frac{d\rho}{d\epsilon}\right)^2 = \rho^2 + \sigma^2 \left(\frac{d\rho}{ds}\right)^2.$$

157. *Rectifying Developable.* A plane drawn through a tangent of a curve, at right angles to the osculating plane, is called a rectifying plane. The rectifying planes of a curve are therefore the tangent planes of a developable surface. It is called the rectifying developable of the curve. Its generating lines are called the rectifying lines of the curve. The curve is a geodesic on its rectifying developable, because its osculating planes are normal planes of this surface, see § 174. Thus if the given curve is drawn on this surface, and the surface is developed into a plane, the curve becomes a straight line. For example, the rectifying developable of a plane curve is the cylinder of which the curve is a cross section; and the rectifying developable of a helix is the cylinder on which this curve is drawn.

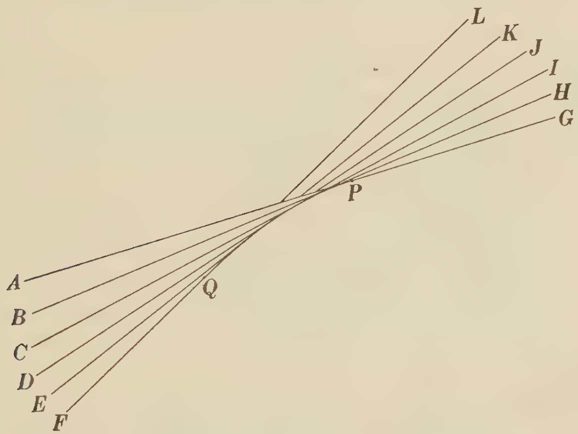
158. *Osculating Cone.* At each point, P, of the curve there is an osculating cone. This is a cone of revolution, with P for vertex, and the rectifying line, PR, for axis, and the tangent PT for a generating line. Thus it osculates the developable of which the tangents of the curve are generating lines, a section of the cone by a plane at right angles to the rectifying line being the circle of curvature of the section of the developable by this plane at their point of contact. If 2α is the vertical angle of the cone, $\tan\alpha = \sigma/\rho$. To prove this, take any point, R, on the rectifying line, and draw perpendiculars, RI, RJ to the osculating planes at P and Q. Then, in the limit, when Q tends to coincidence with P, $RId\epsilon = PId\psi$; and

$$\tan\alpha = \frac{RI}{PI} = \frac{d\psi}{d\epsilon} = \frac{\sigma}{\rho}.$$

159. *Construction of Developable surface.* Let us now return to the consideration of the developable surface, of which the tangents of a curve are the generating lines, and of which the curve is the edge of regression. The diagram given here shows a portion of curve PQ, and six of its tangents, which extend to infinity on each side of their points of contact. It shows that the portion of surface belonging to the arc PQ is bounded by the lines AG and FL, and consists of two sheets, $APQF$ and $GPQL$, which meet at the arc PQ, so that a section of the surface by a normal plane of the curve, at any point of it, has a cusp at that point. The curve is therefore sometimes called the cuspidal edge of the surface. Each tangent lies on both sheets, its point of contact being the point at which it passes from one sheet to the other. If the surface is developed into a plane, the two sheets are in the same plane, and the curve becomes a plane curve with the same curvature, at each point, as that of the given curve.

Accordingly a model of an arc, PQ, of a twisted curve, may be constructed by drawing on paper a plane curve with the required curvature at each point of it, and

cutting away the paper on the concave side of the curve, and then bending the paper so as to give the required torsion. The paper must be bent about the tangents of the curve. But the bending must be done so that two consecutive tangents remain in one plane. This requires that the paper should be bent one way along the tangents drawn from A, B, C, D, E, F to their points of contact, and the opposite way along the continuations of the same tangents from their points of contact to G, H, I, J, K, L. Each of these bendings constructs the curve PQ; but in



one case the portion of paper bent forms only one sheet of the developable, and in the other case it forms only the other sheet. To construct both sheets, two pieces of paper must be used, and one of them must be bent according to one scheme, and the other according to the other scheme. They must then be pasted together, along the curve PQ, in some way which leaves them both free to bend. And when the paper is flattened out into a plane, the two sheets overlap. The actual construction of a model, by means of two pieces of paper, slightly creased along half a dozen tangents, is a simple matter, and shows the arrangement clearly.

In general, any given developable surface has an edge of regression which specifies it. A cone and a cylinder are of course obvious exceptions.

160. The helix may be defined as a curve on the surface of a circular cylinder, which cuts all the generating lines at the same angle. The equations of a helix, on the cylinder $x^2 + y^2 = a^2$, may be written

$$x = a \cos \theta, \quad y = a \sin \theta, \quad z = a\theta \tan \beta,$$

where β is a positive or negative acute angle, which may be called the pitch, and θ is a parameter. These equations give

$$\frac{d\theta}{ds} = \frac{1}{a} \cos \beta, \quad \rho = a \sec^2 \beta, \quad \sigma = -a \sec \beta \operatorname{cosec} \beta.$$

The sign of β settles whether the helix is right- or left-handed. The formulae show that the curvature and torsion are both constant, as is otherwise obvious. The principal normal cuts the axis of the cylinder at right angles. The centre of spherical curvature is in the osculating plane, and the locus of this point is a helix with the same axis as the given helix. The curvature and torsion of a curve, at any point of it, may be specified by the helix which has the same curvature and torsion.

161. *Expansions in ascending powers of s.* The formulae in the previous pages give the differential coefficients of the coordinates, x, y, z, of a point on a curve as follows:

$$\frac{dx}{ds} = l, \quad \frac{dy}{ds} = m, \quad \frac{dz}{ds} = n;$$

$$\frac{d^2x}{ds^2} = \frac{l'}{\rho}, \quad \frac{d^2y}{ds^2} = \frac{m'}{\rho}, \quad \frac{d^2z}{ds^2} = \frac{n'}{\rho};$$

and differentiating again,

$$\frac{d^3x}{ds^3} = \frac{1}{\rho}\frac{dl'}{ds} - \frac{l'}{\rho^2}\frac{d\rho}{ds} = -\frac{1}{\rho}\left(\frac{l}{\rho} + \frac{\lambda}{\sigma}\right) - \frac{l'}{\rho^2}\frac{d\rho}{ds},$$

$$\frac{d^3y}{ds^3} = \frac{1}{\rho}\frac{dm'}{ds} - \frac{m'}{\rho^2}\frac{d\rho}{ds} = -\frac{1}{\rho}\left(\frac{m}{\rho} + \frac{\mu}{\sigma}\right) - \frac{m'}{\rho^2}\frac{d\rho}{ds},$$

$$\frac{d^3z}{ds^3} = \frac{1}{\rho}\frac{dn'}{ds} - \frac{n'}{\rho^2}\frac{d\rho}{ds} = -\frac{1}{\rho}\left(\frac{n}{\rho} + \frac{\nu}{\sigma}\right) - \frac{n'}{\rho^2}\frac{d\rho}{ds}.$$

Let us now take the origin on the curve, and the tangent and principal normal and binormal at the origin for axes of x, y and z, and measure s from the origin. And let us use the suffix 0 to denote that the expressions to which it is applied are given the values which they have at the origin. Then

$$\left(\frac{dx}{ds}\right)_0 = 1, \quad \left(\frac{dy}{ds}\right)_0 = 0, \quad \left(\frac{dz}{ds}\right)_0 = 0;$$

$$\left(\frac{d^2x}{ds^2}\right)_0 = 0, \quad \left(\frac{d^2y}{ds^2}\right)_0 = \frac{1}{\rho_0}, \quad \left(\frac{d^2z}{ds^2}\right)_0 = 0;$$

$$\left(\frac{d^3x}{ds^3}\right)_0 = -\frac{1}{{\rho_0}^2}, \quad \left(\frac{d^3y}{ds^3}\right)_0 = -\frac{1}{{\rho_0}^2}\left(\frac{d\rho}{ds}\right)_0, \quad \left(\frac{d^3z}{ds^3}\right)_0 = -\frac{1}{\rho_0\sigma_0}.$$

Now if (x, y, z) are the coordinates of any point on the curve, regarded as functions of s, Maclaurin's theorem gives the following series in ascending powers of s,

$$x = x_0 + \left(\frac{dx}{ds}\right)_0 s + \frac{1}{2}\left(\frac{d^2x}{ds^2}\right)_0 s^2 + \frac{1}{6}\left(\frac{d^3x}{ds^3}\right)_0 s^3 + \ldots\ldots,$$

$$y = y_0 + \left(\frac{dy}{ds}\right)_0 s + \frac{1}{2}\left(\frac{d^2y}{ds^2}\right)_0 s^2 + \frac{1}{6}\left(\frac{d^3y}{ds^3}\right)_0 s^3 + \ldots\ldots,$$

$$z = z_0 + \left(\frac{dz}{ds}\right)_0 s + \frac{1}{2}\left(\frac{d^2z}{ds^2}\right)_0 s^2 + \frac{1}{6}\left(\frac{d^3z}{ds^3}\right)_0 s^3 + \ldots\ldots;$$

that is to say

$$x = s - \frac{1}{6{\rho_0}^2}s^3 + \ldots\ldots,$$

$$y = \frac{1}{2\rho_0}s^2 - \frac{1}{6{\rho_0}^2}\left(\frac{d\rho}{ds}\right)_0 s^3 + \ldots\ldots,$$

$$z = -\frac{1}{6\rho_0\sigma_0}s^3 + \ldots\ldots$$

The terms here written give, in general, approximate values of the coordinates of a point near the origin. And from them, approximate values of other quantities related to the curve can be obtained; giving a comparison of their relative magnitudes when s is small.

Let D be the length, and $(0, M, N)$ the direction cosines, of the shortest distance between the axis of x and the tangent

at the neighbouring point (x, y, z) on the curve. If (l, m, n) are the direction cosines of this tangent,

$$l = 1 - \frac{1}{2\rho_0{}^2} s^2 + \ldots\ldots, \qquad m = \frac{1}{\rho_0} s - \frac{1}{2\rho_0{}^2}\left(\frac{d\rho}{ds}\right)_0 s^2 + \ldots\ldots,$$

$$n = -\frac{s^2}{2\rho_0\sigma_0} + \ldots\ldots.$$

$$mM + nN = 0, \qquad D = My + Nz, \qquad m^2 + n^2 = \frac{s^2}{\rho_0{}^2} + \ldots\ldots.$$

Therefore

$$D = \pm \frac{-ny + mz}{\sqrt{(m^2 + n^2)}} = \pm \frac{1}{12} \frac{s^3}{\rho_0\sigma_0} + \ldots\ldots.$$

Now the origin may be any point on the curve, so the expression found for D gives the shortest distance between the tangents at any two neighbouring points.

It shows the conditions under which, in a calculation involving infinitesimals, it is correct to treat consecutive tangents as intersecting.

162. *Sign of the Curvature.* For the sake of simplicity, and because it is a common practice, the curvature, $1/\rho$, has been taken to be a positive number; and the positive direction (l', m', n') along the principal normal has been taken to be the direction, along that line, towards the concavity of the curve. This settles the positive direction (λ, μ, ν), taken along the binormal, because the three directions (l, m, n), (l', m', n'), and (λ, μ, ν) are required to be conformable with the directions of the axes of x, y and z.

But this can all be reversed, without any change in the formulae obtained, and with only slight verbal changes in the argument. The numbers whose signs would be reversed are ρ, l', m', n', λ, μ, ν. It will be seen that this change does not affect the fundamental formulae.

The more general and preferable procedure, which would allow for this reversal of signs, without the signs of l, m, n or σ being affected, is to make arbitrary choice of a positive direction (l', m', n'), along the principal normal. We then take

$\delta\psi$, and consequently ρ, positive if this direction is towards the concavity of the curve, and negative if it is towards the convexity. The positive direction of the binormal depends on this choice, but σ is not affected. Let us adopt this procedure. The consequent changes in the argument by which the formulae are established are easily made. For example, in the calculation of ρ, we have to add "If PN is "taken in the opposite direction ρ is negative, therefore the "direction cosines of FG are in the limit

$$-\rho\frac{dl}{ds}, \qquad -\rho\frac{dm}{ds}, \qquad -\rho\frac{dn}{ds};$$

"but PN and FG now have opposite directions, therefore "we get the same result as before, namely

$$l' = \rho\frac{dl}{ds}, \qquad m' = \rho\frac{dm}{ds}, \qquad n' = \rho\frac{dn}{ds}."$$

It is important to be able to treat curvature as being either positive or negative, especially in such a case as the comparison of the curvatures of several curves with a common principal normal line. And it must be noted that the sign of the curvature has no relation to the shape of the curve, but only expresses the relation of the curvature to an arbitrarily chosen direction along the line of the principal normal. Thus its significance is totally different from that of the sign of the torsion, which denotes an important feature of the shape of the curve.

163. The formulae for a change of the independent variable are found as follows. Let θ be a new independent variable, which is to replace s. It may be a parameter, of which the coordinates, x, y, z, of a point on the curve, are given functions; or it may be one of these coordinates. We have

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2,$$

$$l = \frac{dx}{ds} = \left(\frac{ds}{d\theta}\right)^{-1}\frac{dx}{d\theta},$$

therefore by differentiation

$$\frac{dl}{ds} = \frac{d^2x}{ds^2} = \left(\frac{ds}{d\theta}\right)^{-3} \left(\frac{d^2x}{d\theta^2}\frac{ds}{d\theta} - \frac{dx}{d\theta}\frac{d^2s}{d\theta^2}\right),$$

therefore

$$\frac{1}{\rho^2} = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2$$

$$= \left(\frac{ds}{d\theta}\right)^{-4} \left\{\left(\frac{d^2x}{d\theta^2}\right)^2 + \left(\frac{d^2y}{d\theta^2}\right)^2 + \left(\frac{d^2z}{d\theta^2}\right)^2 - \left(\frac{d^2s}{d\theta^2}\right)^2\right\}.$$

Also $\displaystyle \frac{\lambda}{\rho} = m\frac{dn}{ds} - n\frac{dm}{ds} = \left(\frac{ds}{d\theta}\right)^{-3}\left(\frac{dy}{d\theta}\frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\frac{d^2y}{d\theta^2}\right),$

and we have two other similar equations; therefore by squaring and adding we get

$$\frac{1}{\rho^2} = \left(m\frac{dn}{ds} - n\frac{dm}{ds}\right)^2 + \left(n\frac{dl}{ds} - l\frac{dn}{ds}\right)^2 + \left(l\frac{dm}{ds} - m\frac{dl}{ds}\right)^2$$

$$= \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2\right\}^{-3} \left\{\left(\frac{dy}{d\theta}\frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\frac{d^2y}{d\theta^2}\right)^2\right.$$

$$\left. + \left(\frac{dz}{d\theta}\frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\frac{d^2z}{d\theta^2}\right)^2 + \left(\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}\right)^2\right\}.$$

A formula for the torsion is found by transforming the determinant which is given at the end of § 154. Writing s' for $\dfrac{ds}{d\theta}$, s'' for $\dfrac{d^2s}{d\theta^2}$ and s''' for $\dfrac{d^3s}{d\theta^3}$, we get by differentiation

$$\frac{dx}{ds} = \frac{1}{s'}\frac{dx}{d\theta}, \quad \frac{d^2x}{ds^2} = \frac{1}{s'^2}\frac{d^2x}{d\theta^2} - \frac{s''}{s'^3}\frac{dx}{d\theta},$$

$$\frac{d^3x}{ds^3} = \frac{1}{s'^3}\frac{d^3x}{d\theta^3} - 3\frac{s''}{s'^4}\frac{d^2x}{d\theta^2} - \left(\frac{s'''}{s'^4} - 3\frac{s''^2}{s'^5}\right)\frac{dx}{d\theta};$$

and we have corresponding results for y and z. But we know from an elementary property of determinants, namely

$$\begin{vmatrix} a\ , & b \\ c + ka, & d + kb \end{vmatrix} = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix},$$

that for our present purpose we are concerned only with the first term in each of these nine formulae. Thus we get

$$-\frac{1}{\rho^2\sigma}=\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{-3}\begin{vmatrix}\frac{dx}{d\theta}, & \frac{dy}{d\theta}, & \frac{dz}{d\theta}\\ \frac{d^2x}{d\theta^2}, & \frac{d^2y}{d\theta^2}, & \frac{d^2z}{d\theta^2}\\ \frac{d^3x}{d\theta^3}, & \frac{d^3y}{d\theta^3}, & \frac{d^3z}{d\theta^3}\end{vmatrix}.$$

164. *Notes and Examples.*

1. The terminology and formulae of velocities and angular velocities and moving axes, which are employed in dynamics, are applicable to the geometry of twisted curves. We have a set of fixed axes of x, y and z; and a point P is conceived to move along a given curve with unit velocity, carrying with it a set of moving axes, namely the tangent and principal normal and binormal. Thus the moving axes have an angular velocity whose components in the directions PT, PN, PB are $-1/\sigma$, 0, $1/\rho$; so that the resultant angular motion is about a line, PE, in the plane TPB, inclined to PT at an angle whose tangent is $-\sigma/\rho$.

This consideration shows at once the character of a curve for which the ρ/σ has the same value at all points. For such a curve, as P moves along it, the line PE is fixed with reference to the moving axes; therefore its direction cannot change with reference to the fixed axes. And the tangent, PT, is inclined to this direction at a constant angle. Thus the characteristic property of this curve is that all its tangents are inclined at the same angle to a certain fixed direction, and are therefore parallel to the generators of a certain cone of revolution. Accordingly if the lines PE are drawn for all points on the curve, they generate a cylinder; and the curve is a curve drawn on this cylinder so as to cut all the generating lines at the same angle. Such a curve may be called a helix on the cylinder in question. If ρ and σ are both constant the cylinder is circular, and the curve is a helix in the more usual sense of this term.

It is also true that, if the tangent of a curve makes a constant angle with a fixed line, the ratio ρ/σ is constant. To prove this, let α be the constant angle, and the axis of z the fixed line. Then $n=\cos\alpha$, and $n'=\rho\dfrac{dn}{ds}$, therefore $n'=0$. Also $n^2+n'^2+\nu^2=1$, therefore $\nu^2=\sin^2\alpha$. And $\dfrac{n}{\rho}+\dfrac{\nu}{\sigma}=-\dfrac{dn'}{ds}=0$; therefore $\sigma=\pm\rho\tan\alpha$.

The formulae for moving axes can be used for calculating the angle between any straight line related to a point P of a curve, and the corresponding line related in the same way to an adjacent point, Q. (See the *Quarterly Journal of Mathematics*, vol. VII, p. 37.)

2. Show that the curvature and torsion of the curve

$$\frac{x}{1-3\theta} = \frac{y}{1+3\theta} = \frac{z}{6\sqrt{2}\theta^2} = \theta,$$

are in a constant ratio. (S.)

[The question is answered by finding the relation between the direction cosines of the tangent, $l^2 + m^2 - 2\sqrt{2}mn - 2\sqrt{2}nl - 2lm = 0$, see note 1. Further calculation gives $\pm 1/\rho = -1/\sigma = 3\sqrt{2}\,(1 + 18\theta^2)^{-2}$.]

3. A curve is given by the equations

$$x = a\,(\theta + \sin\theta), \quad y = a\,(1 - \cos\theta), \quad z = 4a\cos\tfrac{1}{2}\theta.$$

Prove that $16a^2/\rho^2 = 1 + \cos^2\tfrac{1}{2}\theta$, and that the torsion vanishes when $\sin\tfrac{1}{2}\theta = 0$. (C.)

4. For the curve $x = 6\theta$, $y = 3\theta^2$, $z = \theta^3$, find the curvature, and the torsion, and the radius of spherical curvature, and the equation of the osculating plane. (C.)

[$\pm\,\rho = \sigma = -\tfrac{3}{2}(2 + \theta^2)^2$, $R = \tfrac{3}{2}(2 + \theta^2)^2(1 + 4\theta^2)^{\frac{1}{2}}$, and the equation of the osculating plane is $\theta^2 x - 2\theta y + 2z - 2\theta^3 = 0$.]

5. Prove that for the curve

$$x = 2t\sec\alpha, \quad y = t^2, \quad z = \tfrac{1}{3}t^3 + t\tan^2\alpha,$$

t being a parameter, and α a given angle, $\sigma = \pm\,\rho\cos\alpha$. (C.)

6. Find the radius of curvature of the curve given by

$$x = b\cos\theta, \quad y = b\sin\theta, \quad z = c\cosh\frac{b\theta}{c};$$

and prove that the osculating planes all touch the sphere

$$x^2 + y^2 + z^2 = b^2 + c^2. \qquad \text{(C.)}$$

7. Prove that the radius of curvature of the curve $x = 2a\cos t$, $y = 2a\sin t$, $z = bt^2$ is $2a^{-1}(a^2 + b^2t^2)^{\frac{3}{2}}(a^2 + b^2 + b^2t^2)^{-\frac{1}{2}}$.

8. Prove that for a curve drawn on a sphere

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\sigma\frac{d\rho}{ds}\right) = 0.$$

CHAPTER XII

THE GENERAL THEORY OF SURFACES

165. *Notation.* The notation X, Y, Z will sometimes be used here for current coordinates, in the place of x, y, z. The need for an alternative notation arises when x, y, z are required to represent the coordinates of a point which is restricted to being on a certain surface, so that only two of them are independent variables.

There is a recognised notation for the partial differential coefficients of z, when it is a function of two independent variables, x and y. The partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are denoted respectively by p and q; and $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ by r, s and t.

The partial differential coefficients of a function $F(x, y, z)$, with regard to three independent variables, x, y and z, namely $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$, may be written F_1, F_2, F_3; and the second differential coefficients, $\frac{\partial^2 F}{\partial x^2}$, $\frac{\partial^2 F}{\partial x \partial y}$, $\frac{\partial^2 F}{\partial z\,\partial x}$, ..., may be written, in like manner, F_{11}, F_{12}, F_{31}, ...

If x, y, z are connected by an equation, $F(x, y, z) = 0$, and x and y are chosen as independent variables, p, q, r, s and t can be found by differentiation of this equation with regard to x and y, which gives

$$F_1 + F_3 p = 0, \qquad F_2 + F_3 q = 0,$$

$$F_{11} + 2F_{31}p + F_{33}p^2 + F_3 r = 0,$$

$$F_{12} + F_{23}p + F_{31}q + F_{33}pq + F_3 s = 0,$$

$$F_{22} + 2F_{23}q + F_{33}q^2 + F_3 t = 0.$$

If F_3 is zero, x or y may be taken as the dependent variable.

166. The equation of a surface will be written

$$F(x, y, z) = 0.$$

If this is an algebraical equation, it will be assumed that it is written in the standard form, (§ 19). $F(x, y, z)$ is then a continuous function of x, y and z, which has differential coefficients, and a single value at every point. If $F(x, y, z)$ is not algebraical, it will be assumed that it has similar properties so far as we have occasion to use them.

A point on the surface at which F_1, F_2 and F_3 are all zero, as well as F, is called a singular point. In general a given surface does not possess any singular point, because in general no point exists whose three coordinates satisfy four given equations. A point which is not a singular point is called an ordinary point; and in general it is assumed that any point referred to is an ordinary point.

The vertex of a cone is an example of a singular point; and the line of intersection of a pair of planes is an example of a whole line of singular points. At a singular point the direction (F_1, F_2, F_3), which plays an important part at ordinary points of a surface, does not exist. As in the case of plane curves, there are various types of singular points.

167. *Tangent Plane.* At any point at which F_1, F_2 and F_3 are not all zero, the equation

$$dF = F_1 dx + F_2 dy + F_3 dz$$

gives the rate of increase of F in any given direction at this point. Therefore at any such point, (x, y, z), on the surface $F(x, y, z) = 0$, there are tangent lines, namely the straight lines whose direction cosines, (l, m, n), satisfy the equation

$$F_1 l + F_2 m + F_3 n = 0.$$

They are the lines through this point which are at right angles to the direction (F_1, F_2, F_3), that is to say are in the plane

$$F_1(X - x) + F_2(Y - y) + F_3(Z - z) = 0.$$

Therefore this is the tangent plane at the point (x, y, z). At every point on the surface, which is not a singular point, there is a unique tangent plane and normal.

Now F_1, F_2, F_3 are proportional to p, q, -1. Thus the equation of the tangent plane may be written

$$p(X-x)+q(Y-y)-Z+z=0;$$

and the direction cosines of the normal will be written $-p/k$, $-q/k$, $1/k$, where

$$k^2=1+p^2+q^2.$$

To find how the tangent plane varies when the point of contact is shifted in any direction on the surface, x and y are varied independently, and the variations of z, p and q are given by

$$\begin{aligned} dz &= p\,dx + q\,dy, \\ dp &= r\,dx + s\,dy, \\ dq &= s\,dx + t\,dy. \end{aligned}$$

The direction of the shift depends only on $\dfrac{dy}{dx}$, its direction cosines being proportional to dx, dy, dz, or

$$1, \quad \frac{dy}{dx}, \quad p+q\frac{dy}{dx}.$$

And

$$\frac{dp}{dq}=\frac{r+s\dfrac{dy}{dx}}{s+t\dfrac{dy}{dx}}.$$

From this equation the condition that the surface is a developable can be found. A developable, (§ 22), is a surface the tangent plane of which has only one degree of freedom, as in the familiar cases of a cone and a cylinder. Thus a change of p due to a shift of the point of contact implies a certain change of q, independently of the value of $\dfrac{dy}{dx}$. Therefore the condition that the surface is a developable is expressed by the equation

$$\frac{r}{s}=\frac{s}{t}, \quad \text{or} \quad rt-s^2=0.$$

Such an equation, involving partial differential coefficients, is called the differential equation of the family of surfaces whose characteristic property it expresses. In the present case it is the differential equation of developable surfaces.

Any relation between p and q, not involving any of the coordinates, say $f(p, q) = 0$, represents a certain group of developables. For it gives, by differentiation with regard to x and y,

$$\frac{\partial f}{\partial p} r + \frac{\partial f}{\partial q} s = 0, \qquad \frac{\partial f}{\partial p} s + \frac{\partial f}{\partial q} t = 0;$$

and elimination of $\frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ gives $rt - s^2 = 0$. For example, the equation $p = q$ specifies the surfaces which have the property that the normal at every point is equally inclined to the axes of x and y, that is to say a certain group of cylinders.

168. *Curves drawn on a given surface.* For points on a given surface there are two independent variables. If these are x and y, we regard z, p, q, r, s and t as known at each point of the surface.

For points on a given curve there is one independent variable, and the notation of the previous chapter will be used. Thus

$$l = \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds},$$

$$l' = \rho \frac{dl}{ds}, \qquad m' = \rho \frac{dm}{ds}, \qquad n' = \rho \frac{dn}{ds}.$$

And ρ is positive or negative according as the direction of the principal normal is taken towards the concavity, or towards the convexity of the curve, see § 162.

The letter s will appear here as the arc of a curve only in the form ds, so the use of this letter in two senses, as is the usual practice, will not cause any confusion.

Let P be any point on a curve drawn on a given surface, PT the tangent at P with direction cosines (l, m, n), PN the principal normal with direction cosines (l', m', n'), and

PG the normal of the surface with direction cosines $(-p/k, -q/k, 1/k)$, where k is the positive value of $\sqrt{(1+p^2+q^2)}$. Then p, q, r, s and t are known for each point of the curve; and

$$\frac{dp}{ds} = r\frac{dx}{ds} + s\frac{dy}{ds} = rl + sm,$$

$$\frac{dq}{ds} = s\frac{dx}{ds} + t\frac{dy}{ds} = sl + tm.$$

Also the tangent of the curve is at right angles to the normal of the surface at all points; therefore

$$-pl - qm + n = 0,$$

and $$\frac{d}{ds}(-pl - qm + n) = 0.$$

Also $$l^2 + m^2 + n^2 = 1,$$

and by substitution of $pl + qm$ for n we get

$$(1+p^2)\,l^2 + 2pqlm + (1+q^2)\,m^2 = 1.$$

In the special case of r, s and t being all zero the point is an exceptional one, having some analogy to a point of inflexion in plane geometry. It may be defined as a point at which the direction of the normal is stationary, because $\dfrac{dp}{ds}$ and $\dfrac{dq}{ds}$ are zero for every curve drawn through it on the surface. Thus the meaning of r, s and t being zero is independent of the choice of coordinate axes.

To find the relation between the curvature, $1/\rho$, of the curve, drawn on the surface, and the form of the surface, let ϕ be the angle between PN and PG, then

$$\cos\phi = \frac{1}{k}(-pl' - qm' + n');$$

therefore

$$\begin{aligned}\frac{k\cos\phi}{\rho} &= -p\frac{dl}{ds} - q\frac{dm}{ds} + \frac{dn}{ds}\\ &= \frac{d}{ds}(-pl - qm + n) + l\frac{dp}{ds} + m\frac{dq}{ds}\\ &= l(rl+sm) + m(sl+tm)\\ &= rl^2 + 2slm + tm^2.\end{aligned}$$

By means of the relation between l and m this may also be written

$$\frac{k \cos \phi}{\rho} = \frac{r + 2s\mu + t\mu^2}{1 + p^2 + 2pq\mu + (1 + q^2)\mu^2},$$

where μ is written for the ratio $\frac{m}{l}$, or $\frac{dy}{dx}$, for the curve at the point in question.

These formulae show that, at a given point on the surface, the curvature of the curve is in general determined by the position of its osculating plane; but is zero if r, s and t are zero, unless $\cos \phi$ is also zero, in which case it may have any value.

In the following calculations it will be assumed in general that we are dealing with a point at which r, s and t are not all zero.

169. *Meunier's theorem.* A section of a surface by a plane drawn through the normal at any given point is called a normal section of the surface at that point.

Let $1/\rho'$ be the curvature at P of the normal section through PT, that is to say the section for which $\phi = 0$. Then the formula for ρ gives

$$\rho = \rho' \cos \phi.$$

This result is called Meunier's theorem. It is an extension, to other curved surfaces, of a property of plane sections of a sphere; and can be exhibited by a geometrical construction.

For the corresponding formula for torsion see note 4, in § 235, on the curvature and torsion of a geodesic.

170. *Curvature of a Surface.* The curvatures of the normal sections of a surface at a point P give a specification of the curvature of the surface at that point. These sections are plane curves. Let us take PG as their common principal normal, so that $\cos \phi = 1$.

Take the point P for origin, and PG for axis of z, so that $k = 1$. And let $1/\rho$ be the curvature of the normal

section at this point by a plane inclined to the plane of zx at an angle θ. Then $(\cos\theta, \sin\theta, 0)$ are the direction cosines of the tangent PT of this section. Therefore

$$\frac{1}{\rho} = A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta,$$

where A, H and B are known numbers, namely the values of r, s and t calculated with reference to the coordinate axes here adopted. It will be assumed that they are not all zero.

The form of this equation shows that the value of ρ, in terms of θ, can be found by drawing the pair of conics

$$Ax^2 + 2Hxy + By^2 = \pm 1,$$

or in polar coordinates

$$A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta = \pm\frac{1}{R^2},$$

where R is the length of the central radius inclined to the axis of x at an angle θ. These conics are either a pair of conjugate hyperbolas, or a single ellipse (which may be a circle), or a pair of parallel straight lines. And in each case they give a single real value of R for each value of θ. And ρ is equal to R^2 if it is positive, and is equal to $-R^2$ if it is negative. The conics are hyperbolas, or an ellipse, or a pair of straight lines, according as $AB - H^2$ is negative or positive or zero.

Let us now take the principal axes of the conics for new axes of x and y. Then the equation of the conics takes the form

$$A'x^2 + B'y^2 = \pm 1;$$

and the curvature of the normal section by a plane inclined to the new plane of zx at an angle θ is given by

$$\frac{1}{\rho} = A'\cos^2\theta + B'\sin^2\theta,$$

or

$$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2},$$

where ρ_1 and ρ_2 are the values of ρ for the sections $\theta = 0$

and $\theta = \frac{1}{2}\pi$. These two sections are called the principal normal sections of the surface at P, and $1/\rho_1$, $1/\rho_2$ are called the principal curvatures of the surface at this point. This relation between the curvatures of the normal sections, at any given point, is called Euler's theorem. It shows that the curvature of the surface at a given point is fully specified by the positions and curvatures of the two principal normal sections. And each curvature is positive or negative according as the chosen direction, PG, along the normal of the surface, points in the direction of concavity or convexity of the curve.

The theory of conics shows that the two principal normal sections, at a given point, are the normal sections of greatest and least curvature; except when the curvatures of normal sections are all equal, in which case the pair of conics is a circle, and the point is called an umbilic.

The pair of conics may, of course, be specified by one of them. This is called the Indicatrix of the surface at the point in question. And the point is called an elliptic, or a hyperbolic, or a parabolic point, according as the indicatrix is an ellipse, or a hyperbola, or a pair of parallel straight lines. At an elliptic point ρ has the same sign, either positive or negative, for all the sections. At a hyperbolic point the asymptotes of the indicatrix separate the normal sections for which ρ is positive from those for which it is negative, and are called the inflexional tangents at this point. The curvature of each section by a plane through one of these lines is zero. At a parabolic point the curvature has the same sign for all the sections, except one for which it is zero. The shape of the surface at an elliptic point is synclastic, and the shape at a hyperbolic point is anticlastic, see § 93. Along an asymptotic line, that is a curve each tangent of which is an asymptote, $\tan^2\theta = -\rho_2/\rho_1$, and $1/\rho = 0$.

171. As Euler's theorem deals only with plane curves, it is not necessary to use the formulae for twisted curves.

Take any ordinary point, P, on a surface specified by

an algebraical equation, referred to any axes, and shift the origin to this point. Then the equation of the surface is

$$Lx + My + Nz + ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$+ \text{ terms of the 3rd and higher orders} = 0.$$

And there is a tangent plane at the new origin, namely

$$Lx + My + Nz = 0.$$

Let us now change the directions of the axes, so that the normal at the origin becomes the new axis of z, then the equation of the surface is

$$2z = Ax^2 + 2Hxy + By^2 + 2Fyz + 2Gzx + Cz^2$$
$$+ \text{ terms of the 3rd and higher orders.}$$

Write $r\cos\theta$ and $r\sin\theta$ for x and y respectively, then the equation is

$$\frac{2z}{r^2} = A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta + \dots\dots$$

Let us now confine our attention to the normal section, at the origin, by a plane inclined at an angle θ to the plane of zx. The limiting value of $\frac{2z}{r^2}$, when x, y and z tend to zero in this section, is the curvature of this section at the origin, which will be denoted by $\frac{1}{\rho}$. Therefore

$$\frac{1}{\rho} = A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta,$$

because the limiting value of each of the terms not written down is zero. For example, the first of these terms is $Fy\frac{2z}{r^2}$. Here $\frac{1}{\rho}$ is an algebraical number, positive or negative according as the concavity of the section is towards the positive or the negative direction of the axis of z. It may be zero for certain values of θ; and it is zero for all values of θ if A, H and B are all zero.

This shows that if A, B and H are not all zero, the

curvatures of normal sections at the origin are the same as for the paraboloid, or parabolic cylinder,

$$2z = Ax^2 + 2Hxy + By^2.$$

It is a parabolic cylinder if $H^2 - AB$ is zero. Also we know from conics that, by making proper choice of the directions of the axes of x and y, this equation can be put into the form

$$2z = A'x^2 + B'y^2.$$

And the corresponding equation for the curvature of normal sections is

$$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2},$$

where $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are the curvatures of the sections by the new planes of zx and yz. This is Euler's theorem.

An attempt may be made to construct a surface, represented by an algebraical equation, at each point of which there is a tangent plane and normal, and at some point of which the curvatures do not obey Euler's rule. To examine the sort of way in which such an attempt fails, let us take the surface represented by the equation

$$r^2 = z\,(a + b \sin 4\theta)$$

in cylindrical coordinates, (§ 15), where a and b are positive, and $a > b$. This surface consists of a single sheet, the sections of which by planes through the axis of z are parabolas; and the radius of curvature, at the origin, of the normal section by a plane inclined to the plane of zx at an angle θ, is $\frac{1}{2}(a + b\sin 4\theta)$. If b/a is a small fraction, the surface is hardly distinguishable from a paraboloid of revolution. Now the equation in Cartesian coordinates is

$$x^2 + y^2 = z\left\{a + 4bxy\,\frac{x^2 - y^2}{(x^2 + y^2)^2}\right\};$$

and this equation, written in the standard form, is

$$(x^2 + y^2)^3 = z\,\{a\,(x^2 + y^2)^2 + 4bxy\,(x^2 - y^2)\}.$$

It represents a surface which includes, not only the single sheet of surface generated by parabolas, but also every point on the axis of z. And every point on the axis of z, including the origin, is a singular point. At all other points on the surface the curvatures obey Euler's rule.

On the other hand, Euler's theorem is not restricted to surfaces represented by algebraical equations. For all that is necessary in order that it may apply at a given point on any surface is that, when this point is taken as origin, and the normal for axis of z, it should be possible to write the equation of the surface, throughout some finite portion of it surrounding the point, in the form

$$2z = Ax^2 + 2Hxy + By^2 + \epsilon,$$

where ϵ represents terms such that the limiting value of ϵ/r^2, when the origin is approached in a normal section, is zero, r being defined as before.

172. *Lines of Curvature.* A curve drawn on a surface, so that the tangent at every point of it is in one of the planes of principal normal section at that point, is called a line of curvature of the surface.

Thus if lines of curvature can be drawn, covering a surface, two of them, intersecting at right angles, will pass through each point, if it is not an umbilic, and show the directions of greatest and least curvature of the surface at that point. For example, in the case of a surface of revolution, the lines of curvature are the meridians, and the circles which cut the meridians at right angles.

An alternative definition of a line of curvature is that it is a curve along which successive normals of the surface intersect, so as to be the successive generators of a developable surface. This is the definition which was given by Monge, to whom we owe the first investigation of the theory*. The equivalence of the two definitions is in agreement with the symmetry of the indicatrix, and will be

* *Application de l'analyse à la Géométrie,* by M. Monge.

established here by showing that they give the same results.

Let us begin by adopting the first definition. Taking any coordinate axes, with reference to which the equation of the surface is given, and writing k, as before, for the positive value of $\sqrt{(1 + p^2 + q^2)}$, the curvature of a normal section at a point P is given by the equation

$$\frac{k}{\rho} = rl^2 + 2slm + tm^2,$$

where $(l, m, pl + qm)$ are the direction cosines of the tangent of the curve at this point, and l and m are connected by the equation

$$(1 + p^2)\, l^2 + 2pqlm + (1 + q^2)\, m^2 = 1.$$

Now the principal normal sections at a given point, P, are those for which ρ is a maximum or minimum. Accordingly, the values of $\frac{m}{l}$, $\left(\text{that is to say } \frac{dy}{dx}\right)$, for these two sections, are found by differentiating these equations with regard to l, making $\frac{d\rho}{dl}$ zero, and eliminating $\frac{dm}{dl}$. Thus we get

$$rl + sm + (sl + tm)\frac{dm}{dl} = 0,$$

$$(1 + p^2)\, l + pqm + \{pql + (1 + q^2)\, m\}\frac{dm}{dl} = 0;$$

and finally

$$\frac{rl + sm}{(1 + p^2)\, l + pqm} = \frac{sl + tm}{pql + (1 + q^2)\, m}.$$

This is a quadratic equation for $\frac{m}{l}$, namely

$$\{pqt - (1 + q^2)\, s\}\left(\frac{m}{l}\right)^2 + \{(1 + p^2)\, t - (1 + q^2)\, r\}\frac{m}{l} - pqr + (1 + p^2)\, s = 0.$$

It gives the values of $\frac{m}{l}$ for the two principal normal sections at P, in terms of the coordinates of that point.

We can also get from this procedure a quadratic equation giving the curvatures of these two sections in terms of the coordinates of P. To do this, let us write the equation for m/l in the form of a pair of equations,

$$C\{(1+p^2)\,l + pqm\} = rl + sm,$$
$$C\{pql + (1+q^2)\,m\} = sl + tm.$$

Multiplying the first of these equations by l, and the second by m, and adding, the coefficient of C is found to be unity, and we get

$$C = rl^2 + 2slm + tm^2 = \frac{k}{\rho}.$$

Also the pair of equations can be written

$$\{(1+p^2)\,C - r\}\,l = (s - pqC)\,m,$$
$$(pqC - s)\,l = \{t - (1+q^2)\,C\}\,m;$$

and by elimination of l and m we get

$$\{(1+p^2)\,C - r\}\{t - (1+q^2)\,C\} = (s - pqC)(pqC - s),$$

or

$$(1+p^2+q^2)\,C^2 - \{(1+q^2)\,r - 2pqs + (1+p^2)\,t\}\,C + rt - s^2 = 0.$$

This is a quadratic equation for C, the roots of which are k/ρ_1 and k/ρ_2, where $1/\rho_1$ and $1/\rho_2$ are the curvatures of the two principal normal sections at P. It agrees with the fact that a surface for which $rt - s^2$ is zero, at all points, is a developable; for it is shown here to be a surface for which one of the principal curvatures is zero at every point.

We can now write down the differential equation of the lines of curvature of a given surface. For a line of curvature is a curve for which $\frac{dy}{dx}$, at every point, is the same as for a principal normal section at that point. Therefore it is defined by the equation

$$\{pqt - (1+q^2)\,s\}\left(\frac{dy}{dx}\right)^2 + \{(1+p^2)\,t - (1+q^2)\,r\}\frac{dy}{dx} - pqr + (1+p^2)\,s = 0.$$

Regarding p, q, r, s and t as known functions of x and y, this is the differential equation of the projection, on the plane of xy, of the lines of curvature. The solution of it, assuming it to be real, gives two intersecting systems of curves in the plane of xy. It may be necessary to deal separately with two or more sheets of the surface, for each of which p, q, r, s and t are known functions of x and y.

Writing this differential equation in the form

$$H\left(\frac{dy}{dx}\right)^2 + J\frac{dy}{dx} + K = 0,$$

it is obvious that H, J and K are connected by an identical relation, namely

$$(1 + p^2)\,H - pqJ + (1 + q^2)\,K = 0.$$

It will now be shown that this relation provides for the differential equation representing real lines on the surface, intersecting at right angles. In the first place it gives

$$\begin{aligned}(1 + q^2)(J^2 - 4HK) &= (1 + q^2)\,J^2 + 4H\{(1 + p^2)\,H - pqJ\}\\ &= (qJ - 2pH)^2 + J^2 + 4H^2.\end{aligned}$$

This shows that $J^2 - 4HK$ is positive; therefore the roots, say α and β, of the quadratic equation for $\frac{dy}{dx}$ are real. Also the direction cosines of the two corresponding tangents of the curves on the surface are proportional to

$$1,\ \alpha,\ p + \alpha q \quad \text{and} \quad 1,\ \beta,\ p + \beta q;$$

and the condition that these are at right angles is

$$1 + \alpha\beta + (p + \alpha q)(p + \beta q) = 0,$$

or $$1 + p^2 + (\alpha + \beta)\,pq + \alpha\beta\,(1 + q^2) = 0;$$

and substitution of $-J/H$ for $\alpha + \beta$, and K/H for $\alpha\beta$, gives the identity

$$(1 + p^2)\,H - pqJ + (1 + q^2)\,K = 0.$$

There is however a case in which the equation fails to give any value for $\frac{dy}{dx}$, namely that in which H, J and K are all zero. Accordingly this, (it being assumed that r, s and t

are not all zero), is the condition for a point on the surface being an umbilic. In general it may be written

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t}.$$

An umbilic may be a point to which lines of curvature converge, possibly from all directions. But in any case it is a point for which the indicatrix is a circle, and therefore a point at which the search for directions of maximum and minimum curvature must fail.

Let us now verify that we get the same results from Monge's definition. The normals of the surface, along any curve drawn on it, obviously generate a ruled surface. And a line of curvature is now to be defined as a curve for which the surface so generated is a developable. Taking X, Y, Z as current coordinates, the equations of the normal at the point (x, y, z) are

$$X - x + p(Z - z) = 0,$$
$$Y - y + q(Z - z) = 0.$$

And the equations of the normal at another point, $(x + \delta x, y + \delta y, z + \delta z)$, are

$$X - x - \delta x + (p + \delta p)(Z - z - \delta z) = 0,$$
$$Y - y - \delta y + (q + \delta q)(Z - z - \delta z) = 0.$$

The condition that these two normals intersect is that a point, (α, β, γ), can be found such that

$$-\delta x - p\delta z + \delta p(\gamma - z) - \delta p\delta z = 0,$$
$$-\delta y - q\delta z + \delta q(\gamma - z) - \delta q\delta z = 0;$$

and this point is then the point of intersection. The successive normals, at points along a curve, are the generators of a developable, if, and not unless, they satisfy this test in the limiting case of the variation being infinitesimal; and then the point (α, β, γ) is the centre of curvature of the normal section which contains the tangent of the curve; so that the radius of curvature, ρ, of this section is equal to

$k(\gamma - z)$. Accordingly the condition for the curve being a line of curvature is

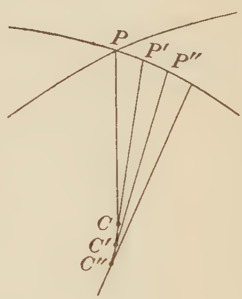
$$-dx - p(pdx + qdy) + (rdx + sdy)\frac{\rho}{k} = 0,$$

$$-dy - q(pdx + qdy) + (sdx + tdy)\frac{\rho}{k} = 0.$$

Writing these equations in the form

$$\frac{\rho}{k} = \frac{(1+p^2) + pq\dfrac{dy}{dx}}{r + s\dfrac{dy}{dx}} = \frac{pq + (1+q^2)\dfrac{dy}{dx}}{s + t\dfrac{dy}{dx}},$$

it will be seen that they are the same equations as those obtained from the former definition. Thus the verification is complete.



The diagram shows the two lines of curvature at a point P; and indicates the generation of a developable by the normals along one of them, namely $PP'P''$, and points, C, C', C'', on the edge of regression of the developable, which are centres of curvature of normal sections.

If we draw all the lines of curvature, and all the developables generated by normals along them, the edges of regression of these developables will trace out two sheets of a surface. This surface is the locus of all the centres of curvature of principal normal sections. It is called the surface of centres. The two sheets of course meet at an umbilic.

173. By Monge's procedure the differential equation of the lines of curvature, of a given surface, can also be found in a form which involves the coordinates symmetrically.

Taking $F(x, y, z) = 0$ as the equation of the surface, the equations of the normal at the point (x, y, z) are

$$X = x + F_1\sigma, \qquad Y = y + F_2\sigma, \qquad Z = z + F_3\sigma,$$

where σ is a parameter, (§ 13), and X, Y, Z are the current

coordinates. Differentiating these equations along a line of curvature, and taking account of the fact that successive normals along this line are to be reckoned as intersecting, so that X, Y, Z are treated as constant, we get

$$\begin{aligned} 0 &= dx + F_1 d\sigma + \sigma dF_1, \\ 0 &= dy + F_2 d\sigma + \sigma dF_2, \\ 0 &= dz + F_3 d\sigma + \sigma dF_3. \end{aligned}$$

This gives, by elimination of σ and $d\sigma$, a relation between x, y and z for points on a line of curvature, namely

$$\begin{vmatrix} dx, & dy, & dz \\ F_1, & F_2, & F_3 \\ dF_1, & dF_2, & dF_3 \end{vmatrix} = 0.$$

Also
$$\begin{aligned} dF_1 &= F_{11}dx + F_{12}dy + F_{13}dz, \\ dF_2 &= F_{21}dx + F_{22}dy + F_{23}dz, \\ dF_3 &= F_{31}dx + F_{32}dy + F_{33}dz. \end{aligned}$$

Thus we have a differential equation in x, y and z, which, in combination with the equation of the surface, gives the lines of curvature.

If it is desired, z can be eliminated between this equation and the equation of the surface. We then get an equation in x and y, which represents, as before, the projection of the lines of curvature on the plane of xy. But the symmetrical form is sometimes preferable.

Let us take, as an example, the quadric surface

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1.$$

Here
$$F_1 = 2\alpha x, \qquad F_2 = 2\beta y, \qquad F_3 = 2\gamma z,$$
$$dF_1 = 2\alpha dx, \qquad dF_2 = 2\beta dy, \qquad dF_3 = 2\gamma dz.$$

And the differential equation is

$$\begin{vmatrix} dx, & dy, & dz \\ \alpha x, & \beta y, & \gamma z \\ \alpha dx, & \beta dy, & \gamma dz \end{vmatrix} = 0,$$

that is to say

$$\left(\frac{1}{\beta} - \frac{1}{\gamma}\right) x\,dy\,dz + \left(\frac{1}{\gamma} - \frac{1}{\alpha}\right) y\,dz\,dx + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) z\,dx\,dy = 0.$$

From this it can be proved that the lines of curvature of a central quadric,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

are the curves of its intersection with its confocals, (§ 144). The equation of a confocal is

$$ax^2 + by^2 + cz^2 = 1,$$

where a, b and c are any numbers such that

$$\frac{1}{a} - \frac{1}{\alpha} = \frac{1}{b} - \frac{1}{\beta} = \frac{1}{c} - \frac{1}{\gamma};$$

that is to say $$\frac{\alpha - a}{\alpha a} = \frac{\beta - b}{\beta b} = \frac{\gamma - c}{\gamma c}.$$

Also $$(\alpha - a) x^2 + (\beta - b) y^2 + (\gamma - c) z^2 = 0,$$

which may be written

$$\alpha a x^2 + \beta b y^2 + \gamma c z^2 = 0,$$

is the equation of a cone drawn through the curve of intersection of the given quadric and the confocal. Therefore, along this curve,

$$ax\,dx + by\,dy + cz\,dz = 0,$$

and $$\alpha a x\,dx + \beta b y\,dy + \gamma c z\,dz = 0.$$

And elimination of a, b and c between the last three equations gives

$$\begin{vmatrix} dx, & dy, & dz \\ \alpha x, & \beta y, & \gamma z \\ \alpha\, dx, & \beta\, dy, & \gamma\, dz \end{vmatrix} xyz = 0.$$

Therefore any curve of intersection of the given quadric with a confocal satisfies the differential equation of a line of curvature; because xyz being zero is a superfluous alternative. And as such curves cover the surface, they include all the lines of curvature.

To integrate this differential equation, we might proceed to eliminate z. But an equivalent course is to adopt the equation given in § 172, namely,

$$H\left(\frac{dy}{dx}\right)^2 + J\frac{dy}{dx} + K = 0,$$

which here takes the form

$$Axy\left(\frac{dy}{dx}\right)^2 + (x^2 - Ay^2 - B)\frac{dy}{dx} - xy = 0,$$

where
$$A = \frac{\gamma - \beta}{\gamma - \alpha}, \qquad B = \frac{\gamma(\beta - \alpha)}{\alpha\beta(\gamma - \alpha)}.$$

And the solution of this, (which was given by Monge), is

$$x^2 - \frac{y^2}{C} = \frac{B}{AC + 1},$$

where C is a constant of integration. Thus the projections of the lines of curvature on the plane of xy are the conics obtained by giving a succession of values to C in this formula.

Consider the line of curvature of the quadric,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

which is the curve of its intersection by the confocal specified by R_1, (§ 145). Along this line R_1 is constant, and we have, (§ 142), the formula

$$R_1 R_2 P = \frac{1}{\alpha\beta\gamma}.$$

Therefore along this line $R_2 P$ is constant. Let D be the length of the semi-diameter of the quadric, or its conjugate, which is parallel to the tangent of the line of curvature, at any point of it; and let p be the length of the perpendicular from the centre on the tangent plane at this point. Then the result obtained is that pD is constant along the line of curvature.

174. *Geodesics.* A curve drawn on a given surface, so that the osculating plane at every point of it contains the normal to the surface at that point, is called a geodesic. It is the curve assumed by a thread stretched on a perfectly smooth surface, because the condition of equilibrium requires that the tangents at two adjacent points of the thread should be in a normal plane. Meunier's theorem, (§ 169), shows that the curve of shortest distance on the surface, between two given adjacent points on the surface, is an element of a normal section of the surface, because

this is the line joining them which has the least curvature. Therefore the line of shortest distance on a surface between two given points is a geodesic.

A geodesic on a developable surface is a curve which becomes a straight line when the surface is developed into a plane.

A geodesic on a sphere is a great circle, (§ 178).

Now the direction cosines of the principal normal of a curve, at a point (x, y, z), are proportional to

$$\frac{d^2x}{ds^2},\quad \frac{d^2y}{ds^2},\quad \frac{d^2z}{ds^2}.$$

Therefore, if $F(x, y, z) = 0$ is the equation of the surface, the condition that the curve is a geodesic is

$$\frac{\dfrac{d^2x}{ds^2}}{F_1} = \frac{\dfrac{d^2y}{ds^2}}{F_2} = \frac{\dfrac{d^2z}{ds^2}}{F_3}, \quad \text{or} \quad \frac{1}{p}\frac{d^2x}{ds^2} = \frac{1}{q}\frac{d^2y}{ds^2} = -\frac{d^2z}{ds^2}.$$

It is assumed here that the curve has an osculating plane, and is therefore not a straight line; and the case of the shortest distance between two points on a surface needs special consideration if it is a straight line.

In the case of a surface of revolution, $z = f(x^2 + y^2)$, the first step in the integration of the equations for a geodesic can be taken as follows. Here

$$p = 2xf'(x^2 + y^2), \qquad q = 2yf'(x^2 + y^2);$$

therefore $$xq - yp = 0.$$

And for a geodesic,

$$\frac{1}{p}\frac{d^2x}{ds^2} = \frac{1}{q}\frac{d^2y}{ds^2} = -\frac{d^2z}{ds^2}.$$

Therefore $$x\frac{d^2y}{ds^2} - y\frac{d^2x}{ds^2} = 0,$$

which gives by integration

$$x\frac{dy}{ds} - y\frac{dx}{ds} = C, \qquad \text{or} \qquad r^2\frac{d\theta}{ds} = C,$$

where $x = r\cos\theta$, $y = r\sin\theta$, and C is a constant along the curve. Now $r\dfrac{d\theta}{ds} = \sin\psi$, where ψ is the angle of intersection of the curve and a meridian of the surface. Therefore $r\sin\psi$ is constant along a geodesic.

For a geodesic on a central quadric,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

we have
$$l = \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds};$$

$$\frac{1}{\alpha x}\frac{dl}{ds} = \frac{1}{\beta y}\frac{dm}{ds} = \frac{1}{\gamma z}\frac{dn}{ds} = \lambda \text{ (say)};$$

$$\alpha x l + \beta y m + \gamma z n = 0.$$

And differentiating the last equation,

$$\alpha x\frac{dl}{ds} + \beta y\frac{dm}{ds} + \gamma z\frac{dn}{ds} + \alpha l^2 + \beta m^2 + \gamma n^2 = 0,$$

therefore

$$\lambda(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2) + \alpha l^2 + \beta m^2 + \gamma n^2 = 0.$$

Let p be the length of the perpendicular from the centre on the tangent plane, and D the semi-diameter of the quadric, or its conjugate, in the direction (l, m, n), then

$$\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2 = \frac{1}{p^2},$$

and
$$\alpha l^2 + \beta m^2 + \gamma n^2 = \frac{1}{R},$$

where R is equal to D^2 or $-D^2$ according as the quadric is an ellipsoid or a hyperboloid. Therefore

$$\lambda\frac{1}{p^2} + \frac{1}{R} = 0.$$

Also

$$\frac{1}{2}\frac{d}{ds}\left(\frac{1}{R}\right) = \alpha l\frac{dl}{ds} + \beta m\frac{dm}{ds} + \gamma n\frac{dn}{ds}$$
$$= \lambda(\alpha^2xl + \beta^2ym + \gamma^2zn) = -\frac{1}{2}\frac{p^2}{R}\frac{d}{ds}\left(\frac{1}{p^2}\right);$$

therefore $\dfrac{1}{p^2}\dfrac{d}{ds}\left(\dfrac{1}{R}\right)+\dfrac{1}{R}\dfrac{d}{ds}\left(\dfrac{1}{p^2}\right)=0$, or $\dfrac{d}{ds}\left(\dfrac{1}{p^2}\times\dfrac{1}{R}\right)=0.$

Accordingly p^2R, and consequently pD, is constant along a geodesic. And as pD is also constant along a line of curvature, it must have the same value for all geodesics that can be drawn tangential to a given line of curvature. It should be noticed that, in the special case of the geodesic being a straight line, λ is zero, and R and D are infinite, and no other result is obtained.

For curvature and torsion of a geodesic see Appendix, § 235, note 4.

175. *The Conoid.* A ruled surface which is generated by a straight line which moves so that it always intersects a given straight line, and is parallel to a given plane, is called a conoid. There are various types of conoids, each of them specified by some further condition which the moving line must satisfy.

The definition shows that a hyperbolic paraboloid is a conoid; because each line belonging to one system of generators intersects any given line belonging to the other system of generators, and is parallel to a certain plane. Here the further condition is that the moving straight line must intersect another given straight line.

If the given straight line, (called the axis), is at right angles to the given plane, the surface is called a right conoid.

The cylindroid, which plays an important part in Sir Robert Ball's theory of screws, is an example of a right conoid. The equation of this surface is

$$z\,(x^2+y^2)-kxy=0;$$

the axis of z being the given straight line, and the plane of xy being the given plane. See example, § 236, 36.

The equation $y = x\tan\lambda z$ represents the conoid which is called a helicoid, and which is exhibited by the smooth under side of a spiral staircase. Any given pair of values of x and y gives an infinite series of values of z.

176. *Notes and Examples.*

1. The quadratic equation, (§ 172), which gives the curvatures, $1/\rho_1$, $1/\rho_2$, of the principal normal sections of a surface, shows that a point on the surface is elliptic or hyperbolic or parabolic according as $rt - s^2$ is positive or negative or zero. The product of the curvatures, namely $1/\rho_1\rho_2$, or $(rt - s^2)/k^4$, is called the measure of curvature of the surface at the point in question. It has been shown by Gauss that, if a surface is bent, without stretching, the measure of curvature is unchanged.

2. The property of a system of confocal quadrics, that their lines of intersection are the lines of curvature on each surface, can be shown to be a general property of systems of surfaces which intersect at right angles. (Dupin's theorem.)

3. Find the equation of the conoid generated by a straight line which moves parallel to the plane of xy, and intersects both the axis of z and the curve

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \qquad x/a + y/b = 1.$$

4. For the surface $xy = cz$, show that the lines of curvature are given by the equation

$$\sqrt{(z^2 + x^2)} \pm \sqrt{(z^2 + y^2)} = \text{constant}. \qquad \text{(S.)}$$

5. Find the coordinates of the umbilics of the surface

$$x^2/a^2 + y^2/b^2 = 2z/c;$$

and prove that a sphere of radius ab/c can be placed so as to touch the surface at the umbilics. (S.)

6. Tangent planes are drawn to the surface $xy = cz$ at the points at which this surface is intersected by the cylinder $x^2 = ay$; show that the edge of regression of the developable which is their envelope is represented by the pair of equations

$$x^2 + 3ay = 0, \qquad x^3 + 27acz = 0.$$

[The equation of a tangent plane is $k^2x + aky - acz = k^3$, where k is a parameter. By differentiating this twice with regard to k we get two more equations. Elimination of k between the first and second gives the equation of the developable; and elimination of k between all three gives the equations of the required curve.]

7. Prove that the planes $y\theta^3 - z\theta^2 + (y + 2x)\,\theta - (2c - z) = 0$, where θ is a parameter, are osculating planes of the curve

$$y : z : y + 2x : 2c - z :: 1 : 3\theta : 3\theta^2 : \theta^3;$$

and that this curve is a geodesic on the quadric surface

$$(x - y)(2x + y) + z(2z - 3c) = 0. \qquad \text{(S.)}$$

8. Find the equation of the developable which contains the conics $xy = 1$, $z = 0$; and $y^2 = 4x$, $z = 1$. And show that its edge of regression lies on the quadric cone $8xy = 9(z-1)^2$. (C.)

9. The axes of two circular cylinders whose radii are a and b intersect at right angles. Show that the curvature of their curve of intersection at a distance z from the plane of the axes is

$$\left\{\frac{a^4}{(a^2-z^2)^3} + \frac{b^4}{(b^2-z^2)^3}\right\}^{\frac{1}{2}}.$$

10. The angular coordinates θ, ϕ, (§ 15), of any point of a curve drawn on a sphere of radius a, are connected by the relation $\sec\theta = \cosh\phi$; prove that the curve cuts the meridian at an angle θ, and that its radius of curvature is $a(1 + 4\cos^2\theta)^{-\frac{1}{2}}$. (C.)

11. A curve on a sphere of radius a cuts all the meridians at an angle β; show that the radius of curvature at the point whose co-latitude and longitude are θ, ϕ, is $a\sin\theta(\sin^2\theta + \sin^2\beta\cos^2\theta)^{-\frac{1}{2}}$.

12. Prove that if θ, ϕ are the angular coordinates, (§ 15), of a point of a curve drawn on a sphere of radius a, and χ is the angle between the osculating plane and the radius, and $1/\rho$, $1/\sigma$, are the curvature and torsion;

i. $\tan\chi = \dfrac{a}{\sin\theta}\dfrac{d}{d\theta}\left(\sin^2\theta\dfrac{d\phi}{ds}\right); \quad \dfrac{1}{\sigma} = \dfrac{d\chi}{ds};$ (C.)

ii. $\dfrac{a^2}{\rho^2} = 1 + \dfrac{a^2}{\sin^2\theta}\left\{\dfrac{d}{d\theta}\left(\sin^2\theta\dfrac{d\phi}{ds}\right)\right\}^2;$ (C.)

iii. $$\frac{a^2}{\rho^2} = 1 + \left\{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right\}^{-3} \times \left\{2\cos\theta\frac{d\phi}{d\theta} + \sin^2\theta\cos\theta\left(\frac{d\phi}{d\theta}\right)^3 + \sin\theta\frac{d^2\phi}{d\theta^2}\right\}^2.$$

13. Show that on the surface formed by the revolution of the circle $x^2 + y^2 = 2ax$ about the axis of y, the geodesics which cut the external equatorial section at an angle $\frac{1}{3}\pi$ touch the circles generated by the revolution of the two points $x = a$, $y = \pm a$. (C.)

14. Prove that the radius of curvature of a geodesic on a cone of revolution varies as the cube of the distance from the vertex.

15. Find the umbilics of the surface $xyz = a^3$; and prove that the principal radius of curvature at an umbilic is $a\sqrt{3}$. (C.)

CHAPTER XIII

SPHERICAL TRIGONOMETRY

177. Spherical trigonometry provides a way of calculating the angles between planes and straight lines, all drawn through one point, by means of the geometry of circles and points on a sphere with its centre at the point in question. The problem is thus reduced to being a two dimensional one, admitting methods of calculation analogous to those of plane trigonometry.

The planes through the centre of the sphere are represented by the circles in which they cut the sphere; and these circles correspond to the straight lines of plane trigonometry. The angles of intersection of the planes are the angles of intersection of these circles, and correspond to the angles of plane trigonometry. And straight lines drawn from the centre of the sphere are represented by the points in which they cut the sphere.

This method has special developments for dealing with the problems of astronomy and navigation. But the fundamental formulae have more general applications, being often simpler than those which are given by coordinate geometry. Those which involve a right angle are particularly simple and convenient.

178. *Definitions.* Any two points at the extremities of a diameter of the sphere are said to be antipodal to each other. Sections of the sphere by planes through the centre are called great circles. Sections by other planes are called small circles. For every great circle there are two antipodal points which are called its poles; they are the extremities of the diameter which is at right angles to its plane.

Through any two antipodal points, any number of great circles can be drawn. Through any two points on the sphere, which are not antipodal, a single great circle can be drawn; namely the section of the sphere by the single plane which contains the radii to these two points.

From any point, A, on the sphere, draw great circle arcs, AB and AC; and let AB', AC' be the tangents to these arcs, drawn in the same directions. Then the positive angle $B'AC'$, less than π, between these tangents, will be called the angle, BAC, between the arcs AB and AC. It is obviously one of the two positive angles, less than π, between the planes of the given arcs AB and AC. A positive angle between π and 2π, between the same arcs, can be similarly defined if we have occasion to use it.

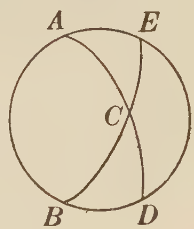
The length of a great circle arc is measured, in spherical trigonometry, by the angle which it subtends at the centre of the circle. This is reckoned as a positive angle between 0 and 2π. The length of the arc, if measured in units of length, is of course the product of this angle and the radius of the sphere. The adoption of an angular measurement of this length makes it unnecessary to define the size of the sphere. If it is necessary to call attention to this, a length so measured may be called an angular length.

The distance between two given points on the sphere is defined, for our present purpose, as being the length, (that is to say the angular length), of a great circle arc joining them. The distance between two antipodal points is the same by all great circle routes, namely π. Between two points which are not antipodal there are two distances, one greater and one less than π; because the whole length of the great circle through them is 2π. But the distance between two points is usually understood to mean the shorter of these two distances, when nothing is said to the contrary.

A great circle arc of length $\frac{1}{2}\pi$ is called a quadrant. Thus every great circle arc drawn from a point on a great circle to one of its poles is a quadrant.

179. *Spherical Triangle.* Take any three points, A, B, C, on the sphere, such that they are not all on one great circle; and draw the great circle arcs BC, CA, AB, each of them less than π. These arcs can always be drawn, and in only one way. The case of two of the points being antipodal is excluded, because in this case the three points would all lie on a great circle. The figure, ABC, formed by these arcs, is called a spherical triangle.

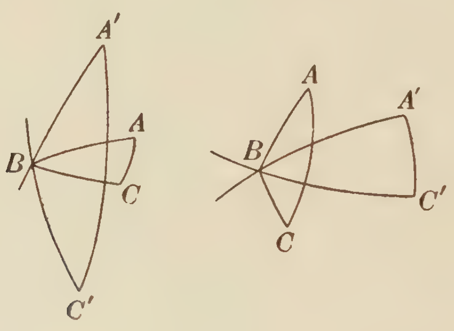
The three arcs are called the sides of the triangle, and their lengths, BC, CA, AB, will be denoted by a, b, c. And the angles, CAB, ABC, BCA, between these arcs, are called the angles of the triangle, and will be denoted by A, B, C. Produce the arc AC to the point D, antipodal to A, and the arc BC to the point E, antipodal to B; and complete the great circle $ABDEA$. Then the diagram shows the angles A, B, C, formed by the intersection of arcs, so that each is less than π. Accordingly the six elements of a triangle, namely A, B, C, a, b, c, are each of them between 0 and π.



The three great circles, when completed, form several other figures, for example the figure $AEDBC$, which is shown in the diagram. But as it is part of the definition of a spherical triangle in this theory, (which we may call the standard theory), that its sides are all less than π, we get only one spherical triangle from three given points.

A more general theory has been constructed, and is used for some purposes in Astronomy, in which this restriction is abandoned. But for most purposes the old established restricted theory, which is adopted here, is more useful. In this theory, if we are concerned with such a figure as $AEDBC$, its properties can be derived from those of the triangle ABC. This particular figure can be specified by angles $\pi - A$, $\pi - B$, $2\pi - C$, and sides a, b, $2\pi - c$.

180. *Polar Triangle.* Take any spherical triangle ABC. Let A' be the pole of the great circle BC which is on the same side of it as A; and let B' be the pole of the great circle CA which is on the same side of it as B; and let C' be the pole of the great circle AB which is on the same side of it as C. Then the spherical triangle $A'B'C'$ is called the polar triangle of the triangle ABC. The construction specifies the triangle without ambiguity, and provides that the arcs AA', BB', CC' are each of them less than $\frac{1}{2}\pi$, and that the arcs AB', AC', BC', BA', CA', CB' are each of them equal to $\frac{1}{2}\pi$. Now this is all that is necessary to secure that ABC is the polar triangle of $A'B'C'$. Thus the two triangles have this reciprocal relation; each is the polar triangle of the other. Let the angles and sides of the triangle $A'B'C'$ be A', B', C', a', b', c'.

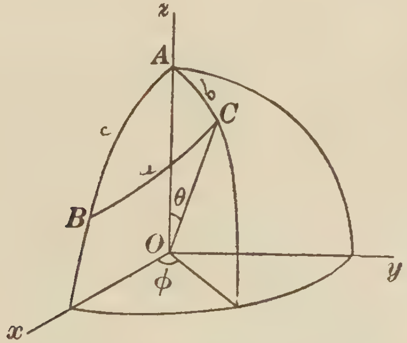


Let us now make a slightly different statement of the construction of the triangle $A'B'C'$ from the triangle ABC. Through B draw an arc at right angles to BC. A' is on this arc, at distance $\frac{1}{2}\pi$ from B, and on the side of BC towards A. Also draw an arc, through B, at right angles to BA. C' is on this arc, at distance $\frac{1}{2}\pi$ from B, and on the side of BA towards C. There are two cases, as shown in the diagram, one with the angle B acute, and the other with the angle B obtuse. In each case the sum of two right angles, $A'BC$ and $C'BA$, is identical with the sum

of the two angles ABC and $A'BC'$. But the angle $A'BC'$ is equal to the length of the arc $A'C'$, that is to say is equal to b'; therefore $B + b'$ is equal to π. From this we can infer the same result for all expressions of this type. That is to say, $A + a'$, $B + b'$, $C + c'$, $A' + a$, $B' + b$, $C' + c$ are each of them equal to π.

This shows that if a spherical triangle exists with angles A, B, C and sides a, b, c, another spherical triangle exists with angles $\pi - a$, $\pi - b$, $\pi - c$, and sides opposite to these angles equal to $\pi - A$, $\pi - B$, $\pi - C$. So that if a general formula is established connecting A, B, C, a, b, c, the formula which is obtained by substituting for these six numbers $\pi - a$, $\pi - b$, $\pi - c$, $\pi - A$, $\pi - B$, $\pi - C$ respectively, is also true. Formulae obtained in this way may be referred to as being obtained by means of the polar triangle.

181. *Relations between sides and angles.* The formulae connecting the six elements, A, B, C, a, b, c, of a spherical



triangle, ABC, will now be investigated. The need to take account of various cases, according to whether sides or angles are greater or less than $\frac{1}{2}\pi$, can be avoided by using direction cosines. Take rectangular axes such that the origin, O, is the centre of the sphere; and OA is the positive direction of the axis of z; and B is in the plane of zx, and

on the positive side of the plane of yz. Then C may be any point on the sphere, not in the plane of zx.

Thus the direction cosines of OA are $(0, 0, 1)$, and the direction cosines of OB are $(\sin c, 0, \cos c)$. Let θ, ϕ be the angular polar coordinates of C, § 15; then the direction cosines of OC are

18. OP = OC here

$$\sin\theta\cos\phi, \quad \sin\theta\sin\phi, \quad \cos\theta.$$

here $r = 1$ see p 18

Therefore the formula for the cosine of the angle between two given directions, (§ 6), gives

$\cos\phi = ll' + mm' + nn'$

$$\cos a = \sin c \sin\theta\cos\phi + \cos c\cos\theta.$$

But $\theta = b$; and A is either ϕ or $2\pi - \phi$, whichever is less than π. Therefore

$$\sin\theta = \sin b, \quad \cos\theta = \cos b, \quad \text{and} \quad \cos\phi = \cos A.$$

Therefore $\quad \cos a = \cos b\cos c + \sin b\sin c\cos A.$

The diagram given here shows the simple case in which $c < \frac{1}{2}\pi$, and C is in the positive octant of the axes. In the calculation, care has been taken to provide for all cases.

Similarly
$$\cos b = \cos c\cos a + \sin c\sin a\cos B,$$
$$\cos c = \cos a\cos b + \sin a\sin b\cos C.$$

Hence we get, by means of the polar triangle,

$$-\cos A = \cos B\cos C - \sin B\sin C\cos a,$$
$$-\cos B = \cos C\cos A - \sin C\sin A\cos b,$$
$$-\cos C = \cos A\cos B - \sin A\sin B\cos c.$$

From these equations, together with the restriction imposed on the sides and angles, of being between 0 and π, a number of other formulae can be derived.

From the first equation we get

$$\sin^2 A = 1 - \frac{(\cos a - \cos b\cos c)^2}{\sin^2 b\sin^2 c}.$$

Therefore

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}{\sin^2 a\sin^2 b\sin^2 c}.$$

Therefore $\frac{\sin A}{\sin a}$, being positive, is equal to the positive square root of the expression on the right hand side of this equation. But this is symmetrical in a, b and c, therefore $\frac{\sin B}{\sin b}$ and $\frac{\sin C}{\sin c}$ have the same value, and we get the formulae

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Here the polar triangle does not supply any new result.

Each of the equations which have been obtained connects four elements, either angles or sides, of the triangle. The only combination of four elements which they do not provide for is a set of four elements adjacent to one another, two angles and two sides. A formula connecting A, b, C, a will provide for any such combination. This may be obtained from the following equations see McL. p. 59

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\sin c \sin A = \sin a \sin C.$$ from "sin Rule"

Substituting, in the first of these equations, the value of $\cos c$ given by the second equation, and the value of $\sin c$ given by the third, and dividing throughout by $\sin a \sin b$, we get

1. $$\cot A \sin C + \cos b \cos C = \cot a \sin b.$$

From this we can write down five other formulae of the same type, namely:

2. $$\cot A \sin B + \cos c \cos B = \cot a \sin c,$$ inwards →

3. $$\cot B \sin A + \cos c \cos A = \cot b \sin c,$$ see McL. p.60

4. $$\cot B \sin C + \cos a \cos C = \cot b \sin a,$$

5. $$\cot C \sin B + \cos a \cos B = \cot c \sin a,$$

6. $$\cot C \sin A + \cos b \cos A = \cot c \sin b.$$

From any one of these, the polar triangle gives another formula of the same series, and no independent result.

Half angles and half sides of a spherical triangle, being between 0 and $\frac{1}{2}\pi$, give formulae which, in consequence of this restriction, do not involve any ambiguities of sign. We have

$$\sin^2 \tfrac{1}{2}A = \tfrac{1}{2}(1 - \cos A) = \tfrac{1}{2}\left(1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)$$

$$= \frac{\cos (b - c) - \cos a}{2 \sin b \sin c} = \frac{\sin \frac{1}{2}(a - b + c) \sin \frac{1}{2}(a + b - c)}{\sin b \sin c};$$

and this result may be written

$$\sin \tfrac{1}{2}A = \sqrt{\left\{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}\right\}},$$

where $2s = a + b + c$, and the symbol of square root denotes the positive value of the square root. Similarly

$$\cos^2 \tfrac{1}{2}A = \tfrac{1}{2}(1 + \cos A) = \frac{\cos a - \cos (b + c)}{2 \sin b \sin c}$$

$$= \frac{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(-a + b + c)}{\sin b \sin c}.$$

Therefore $$\cos \tfrac{1}{2}A = \sqrt{\left\{\frac{\sin s \sin (s - a)}{\sin b \sin c}\right\}}.$$

Therefore $$\tan \tfrac{1}{2}A = \sqrt{\left\{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}\right\}},$$

and

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\{\sin s \sin (s - a) \sin (s - b) \sin (s - c)\}}.$$

The expressions thus obtained for sines and cosines of half angles in terms of the sides give

$$\sin \tfrac{1}{2}(B + C) = \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C + \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C$$

$$= \sqrt{\left\{\frac{\sin s \sin (s - a)}{\sin b \sin c}\right\}} \cdot \frac{\sin (s - c) + \sin (s - b)}{\sin a}$$

$$= \cos \tfrac{1}{2}A \frac{\sin \frac{1}{2}(2s - b - c) \cos \frac{1}{2}(b - c)}{\sin \frac{1}{2}a \cos \frac{1}{2}a}$$

$$= \cos \tfrac{1}{2}A \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}a}.$$

This may be written

$$\frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}A} = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}a},$$

and by an exactly similar procedure we get

$$\frac{\sin \frac{1}{2}(B-C)}{\cos \frac{1}{2}A} = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}a},$$

$$\frac{\cos \frac{1}{2}(B+C)}{\sin \frac{1}{2}A} = \frac{\cos \frac{1}{2}(b+c)}{\cos \frac{1}{2}a},$$

$$\frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2}A} = \frac{\sin \frac{1}{2}(b+c)}{\sin \frac{1}{2}a}.$$

The second of these formulae shows that $B-C$ and $b-c$ are either both positive, or both negative, or both zero. That is to say, if we take two angles and the sides opposite to them, the greater angle is opposite to the greater side. Also the third formula shows that $B+C$ is greater or less than π according as $b+c$ is greater or less than π.

Combining these formulae we get

$$\tan \tfrac{1}{2}A = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2}(B+C) = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \tfrac{1}{2}(B-C).$$

Or this may be obtained directly from the expression for $\tan \frac{1}{2}A$ in terms of the sides. Similarly another combination gives

$$\tan \tfrac{1}{2}a = \frac{\cos \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b+c) = \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b-c).$$

From these results we get several others by means of the polar triangle, namely

$$\sin \tfrac{1}{2}a = \sqrt{\left\{-\frac{\cos S \cos (S-A)}{\sin B \sin C}\right\}},$$

where $2S = A + B + C$,

$$\cos \tfrac{1}{2}a = \sqrt{\left\{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}\right\}},$$

$$\tan \tfrac{1}{2}a = \sqrt{\left\{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right\}},$$

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}}.$$

Or these might be proved directly by the procedure

$$\sin^2 \tfrac{1}{2}a = \tfrac{1}{2}(1 - \cos a) = \tfrac{1}{2}\left(1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C}\right),$$

and so on.

The chief interest of some of these results is their similarity to formulae of plane trigonometry.

182. *Right angled triangles.* From the formulae for a general triangle, those for a right angled triangle can be derived. Taking C as the right angle, we get at once six distinct formulae, each connecting three elements of the triangle. These are usually and conveniently written

$$\cos c = \cos a \cos b, \qquad \cos c = \cot A \cot B,$$

$$\sin a = \sin c \sin A, \qquad \sin a = \cot B \tan b,$$

$$\cos A = \sin B \cos a, \qquad \cos A = \tan b \cot c.$$

And we get four more, which are practically repetitions of these, by interchanging A, B and a, b, namely

$$\sin b = \sin c \sin B, \qquad \sin b = \cot \mathrm{A} \tan a,$$

$$\cos B = \sin A \cos b, \qquad \cos B = \tan a \cot c.$$

As a and b may be quadrants, we must be prepared to write the equation $\sin a = \cot B \tan b$ in the form

$$\sin a \cot b = \cot B,$$

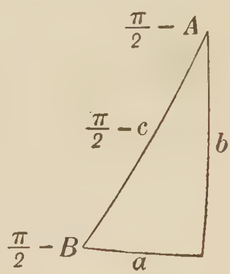
and to make a similar change in the other equations which involve $\tan a$ or $\tan b$. Bearing this in mind we may refrain from disturbing the convenient symmetry of this list of equations.

The polar triangle of a right angled triangle has a quadrant for one of its sides, and is called a quadrantal triangle. Thus we get the formulae for a quadrantal triangle, of which the side c is a quadrant, by substituting $\pi - a$, $\pi - b$, $\pi - A$, $\pi - B$, $\pi - C$ for A, B, a, b, c respectively

in the formulae for a right angled triangle. This substitution gives

$$\begin{aligned}
\cos C &= -\cos A \cos B, & \cos C &= -\cot a \cot b,\\
\sin A &= \sin C \sin a, & \sin A &= \cot b \tan B,\\
\cos a &= \sin b \cos A, & \cos a &= -\tan B \cot C,\\
\sin B &= \sin C \sin b, & \sin B &= \cot a \tan A,\\
\cos b &= \sin a \cos B, & \cos b &= -\tan A \cot C.
\end{aligned}$$

183. *Napier's Rules.* The first person who called attention to the symmetry of the equations for a right angled triangle was Napier, the inventor of logarithms. He published in 1614 what are called Napier's rules of circular parts. Napier's circular parts of a right angled triangle, ABC, with C for the right angle, are a, $\frac{\pi}{2} - B$, $\frac{\pi}{2} - c$, $\frac{\pi}{2} - A$, b, arranged round the triangle as shown in the diagram. Any one of these may be called the middle part; this has two adjacent parts, one on each side of it, and the other two are called its opposite parts. And the ten formulae are all given by one rule, namely:



sine of middle part

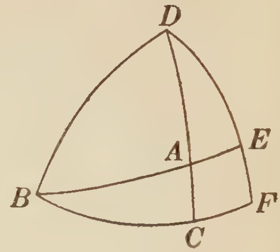
= product of cosines of opposite parts

= product of tangents of adjacent parts.

It is a help, in remembering this rule, to notice that the vowels, i, o and a, indicate which parts the sine, cosine and tangent belong to, namely middle, opposite and adjacent.

For triangles with sides not greater than $\frac{1}{2}\pi$, the uniformity of this rule can be proved as follows. Draw a triangle, DBF, with all its sides and angles equal to $\frac{1}{2}\pi$; and draw any two intersecting quadrants, DAC and BAE,

so that we have a triangle ABC, and a second triangle, ADE, related to it. The circular parts of the first triangle, ABC, taken clockwise from the right angle, C, are



$$a, \quad \tfrac{1}{2}\pi - B, \quad \tfrac{1}{2}\pi - c, \quad \tfrac{1}{2}\pi - A, \quad b.$$

And those of the second triangle, ADE, taken in the same way, are

$$\tfrac{1}{2}\pi - c, \quad \tfrac{1}{2}\pi - A, \quad b, \quad a, \quad \tfrac{1}{2}\pi - B.$$

These are the same as for the first triangle, in the same cyclical order, but starting with $\tfrac{1}{2}\pi - c$. Similarly we get a third triangle from the second, with the same parts, in the same cyclical order, starting from b; and a fourth with the parts starting from $\tfrac{1}{2}\pi - B$; and a fifth with the parts starting from $\tfrac{1}{2} - A$. This is all; because a sixth triangle, derived from the fifth, would have the same parts starting from a, and would therefore be the same as ABC. Napier explained this scheme by drawing all the five triangles, making an ingenious but comparatively complicated figure. The result obtained from them is that, if the rule is true when any one part is chosen as the middle part, it is true in all cases.

184. *Solution of triangles.* If any three of the six elements, A, B, C, a, b, c, of a general spherical triangle are given, we have equations for finding the other three. As each element is between 0 and π, it is specified without ambiguity by its cosine. Therefore if either the three sides, or the three angles, or two sides and the included angle, or two angles and the side between them, are given, the remaining sides and angles are given without ambiguity by the first set of six formulae of § 181. But if the data are two sides and the angle opposite to one of them, or two angles and the side opposite to one of them, the solution of the equations for finding the remaining elements may involve ambiguity.

These two cases are practically only one, because each is obtainable from the other by means of the polar triangle. Let the angle B and the sides b and c be given. In the first place we have the case in which B, b and c are each of them $\frac{1}{2}\pi$, so that A and a are equal and may have any value. Excluding this particular case, the equation

$$\sin C = \frac{\sin c}{\sin b} \sin B$$

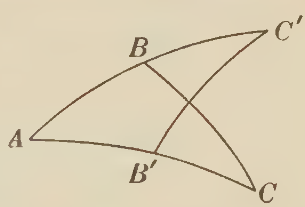
gives two values for C, supplementary to one another, except when it gives $C = \frac{1}{2}\pi$. This is like the ambiguous case in the solution of plane triangles. Either both the values of C are consistent with the data, or only one of them. For a given value of C, A and a are given by the equations

$$\tan \tfrac{1}{2}A = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2}(B+C),$$

$$\tan \tfrac{1}{2}a = \frac{\cos \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b+c).$$

It is assumed here that the data are possible data for a triangle.

185. *Right and Left handed screws.* If the sides of a spherical triangle, and consequently the angles opposite to them, are all known, it is, in general, necessary to settle also the order in which the sides are arranged before the triangle is fully specified. For example, if ABC is a triangle with the given sides and angles, there is also a triangle $AB'C'$ with the same sides and angles, distinct from ABC, except when AB and AC are equal. The difference between these triangles may be expressed by saying that the points ABC are arranged clockwise, and the points $AB'C'$ anticlockwise. This is a description which implies a knowledge of whether the sphere is looked at from the outside,



as is usually intended when a diagram is drawn in spherical trigonometry, or from the inside, as when we look at stars in the sky. The difference is seen when we compare a chart of a portion of the sky with a celestial globe. If ABC is any spherical triangle, and α, β, γ are the points antipodal to A, B, C, the triangles ABC and $\alpha\beta\gamma$ are said to be antipodal to each other. They have the same sides and angles, but one is arranged clockwise and the other anticlockwise, so they are not in general identical.

We are here dealing with a distinction, of frequent occurrence in geometry and physics, which is usually specified by reference to the distinction between a right handed screw and a left handed screw. The direction in which an ordinary corkscrew moves in a fixed cork is said to make a right handed screw with the way in which the corkscrew is turned. Accordingly the direction from the centre of the earth to the north pole is said to make a right handed screw with the direction of the earth's diurnal rotation.

Sir Robert Ball*, in order to obtain a more concise terminology, which at the same time should be easily intelligible, has suggested that the reference to a right handed screw should be replaced by a reference to the relation of the earth's north pole to the earth's diurnal rotation. He calls the north pole the "*nole*" of this rotation, and the south pole the "*antinole*." The word nole is easily connected with north pole without risk of error.

This word may similarly be used to distinguish between the two opposite directions along any line which threads a given circuit, along which there is any circulation of known direction. For example, the direction of magnetization of an electro-magnet can be specified by saying that it is the nole of the electric current. The way the coordinate axes are drawn in the diagrams in this book may be specified by saying that axis of z is the nole of a rotation about it from Ox towards Oy. It would be a convenience

* *Spherical Astronomy*, p. 25.

to have a single word with this meaning. However the usual terminology is a reference to a right handed screw; and it must be admitted that there is nothing else which is so easily and universally intelligible.

186. In the limiting case in which the radius of the sphere becomes infinite, in comparison with the linear dimensions of spherical triangles drawn on it, we get plane geometry, in which arcs of great circles become straight lines, their radii becoming infinite, while small circles are still circles. Thus, in this limiting case, all the formulae of spherical trigonometry become formulae of plane trigonometry, a fact which occasionally supplies a useful test of their correctness. Angles between arcs of great circles become angles between the corresponding straight lines. And the arcs themselves, measured by the angles which they subtend at the centre of the sphere, become infinitesimal angles whose limiting ratios to one another are equal to the ratios of the corresponding lengths in the plane.

Take any formula of spherical trigonometry, and substitute λa, λb, λc for a, b, c respectively. Then in the limiting case in which λ tends towards zero, the formula must become either an identity, or else a formula for a plane triangle, of which a, b, c, A, B, C are the sides and angles. Thus the formulae which involve ratios of sines of arcs become

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{bc},$$

$$\tan \tfrac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \tfrac{1}{2}A,$$

and so on. The formula

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a$$

gives at once, for plane trigonometry,

$$-\cos A = \cos B \cos C - \sin B \sin C,$$

which affords a verification of the signs in the spherical

trigonometry formula. The first of the cotangent formulae, (§ 181), may be written

$$\cot A \sin C + \cos b \cos C = \cos a \frac{\sin B}{\sin A},$$

and this gives at once for plane trigonometry

$$\cos A \sin C + \cos C \sin A = \sin B,$$

which may serve as a test of the formula in question.

Take the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Here the corresponding formula of plane trigonometry is given by the limiting case, when λ tends to zero, of

$$1 - \tfrac{1}{2}\lambda^2 a^2 + \ldots = (1 - \tfrac{1}{2}\lambda^2 b^2 + \ldots)(1 - \tfrac{1}{2}\lambda^2 c^2 + \ldots)$$
$$+ (\lambda b - \ldots)(\lambda c - \ldots) \cos A,$$

that is to say $\quad a^2 = b^2 + c^2 - 2bc \cos A.$

Various propositions about transversals, in plane geometry, are particular cases of propositions about the ratios of sines of arcs of great circles in spherical trigonometry.

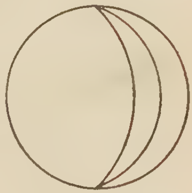
187. *Solid angles, and Areas.* The term "solid angle" may be defined as follows. With a given point for centre let a sphere of unit radius be drawn; and let radii be drawn, from this point, through all points of the boundary of a given area on the sphere. Then these radii are said to enclose a solid angle whose magnitude is that of the given area. The radii may be drawn according to any given rule which provides that they enclose a definite area on the sphere. Thus a solid angle is often specified as being that which is subtended at the given point by some circuit, or area, which is not on the sphere; the radii are then to be drawn to all points on this circuit, or on the boundary of the area. It is only for the purpose of the measurement of the solid angle that the sphere of unit radius is needed.

Two solid angles are considered to be equal if they have the same magnitude, irrespective of their shapes.

The sum of all the solid angles that can be drawn at a given point, without overlapping, is equal to 4π, this being the area of a sphere of unit radius. A set of rectangular coordinate planes encloses eight equal solid angles at the origin, so the magnitude of each of these is $\frac{1}{2}\pi$.

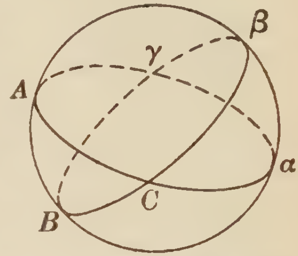
Let r be the radius of the sphere used in spherical trigonometry. Then the whole area of this sphere is $4\pi r^2$; and any given area on this sphere is ωr^2, where ω is the magnitude of the solid angle subtended by this area at the centre of the sphere.

A spherical triangle is a particular case of a spherical polygon whose sides are arcs of great circles. Spherical polygons of three or more sides correspond to the polygons of plane geometry. But we also have, on the sphere, the polygon of one side and the polygon of two sides. Any complete great circle is a polygon of one side. A polygon of two sides is called a lune. The angular points of a lune must be two antipodal points in which two great circles intersect. Thus the sides of a lune are both semicircles, and its two angles are equal. Accordingly a lune is specified by this angle, which is called the angle of the lune. By drawing a number of the great circles which have two common points of intersection, we get a set of lunes which exactly cover the sphere; and by making them all equal, we find that the ratio of the area of a lune of angle θ to the area of the sphere is $\frac{\theta}{2\pi}$. Thus the area of a lune of angle θ is $2\theta r^2$. This is the key to finding other areas bounded by great circles. It should be noticed that the area of any triangle is equal to the area of its antipodal triangle, because these two triangles have the same sides and the same angles opposite to them.



To find the area, Δ, of a given spherical triangle ABC, draw the three great circles of which the sides of the triangle are arcs, and let α, β, γ be the points which are antipodal to

A, B, C respectively. The diagram shows three lunes whose angles are A, B, C, and whose areas are therefore $2Ar^2$, $2Br^2$ and $2Cr^2$. Also we know the area of a hemisphere, namely $2\pi r^2$. Now the hemisphere which is bounded by the circle $AB\alpha\beta$ and contains the point C is made up of the lune of area $2Ar^2$, and the area $AC\beta$ which is $2Br^2 - \Delta$, and the area $C\alpha\beta$ which is $2Cr^2 -$ area $\alpha\beta\gamma$. But $\alpha\beta\gamma$ is the triangle antipodal to ABC, and therefore has the same area Δ, (§ 185). Accordingly we have the equation



$$2\pi r^2 = 2Ar^2 + 2Br^2 - \Delta + 2Cr^2 - \Delta,$$

or

$$\Delta = (A + B + C - \pi)\, r^2.$$

$A + B + C - \pi$ is called the spherical excess of the triangle.

The area of any spherical polygon is easily found, in terms of its angles, by dividing it into triangles.

188. *Notes and Examples.*

1. By means of the procedure of § 186, some approximate results can be obtained which are used in terrestrial surveying.

The most important of these, which is called Legendre's theorem, is that, if the sides of a spherical triangle are small compared with the radius of the sphere, each angle exceeds by one-third of the "spherical excess" the corresponding angle of the plane triangle, the sides of which are of the same length as the arcs of the spherical triangle. In this calculation the spherical excess is of the order of the square of the ratio of a side of the triangle to the radius of the sphere, and higher powers of this ratio are neglected.

2. The angles of the triangles which are the faces of a tetrahedron are given. Find the angles between the faces, and the inclination of an edge to each of the faces in which it does not lie. (C.)

3. Prove that the arcs drawn from the angles of a spherical triangle perpendicular respectively to the opposite sides are concurrent.

4. A ship, starting in north latitude l, sails uniformly along a great circle to a place in latitude l_3, without crossing the equator, and is in latitude l_2 halfway through the voyage. Prove that if x is the ratio of the distance traversed to the radius of the earth, assumed to be a sphere,

$$\cos \tfrac{1}{2}x = \frac{\sin l_1 + \sin l_3}{2 \sin l_2};$$

and calculate the change of longitude.

5. In a right-angled spherical triangle one angle is 46°, and the opposite side is 42°; sketch a diagram showing how to construct two different triangles with these data.

6. Prove that for a spherical triangle whose area is one-eighth of the surface of the sphere

$$\sin 2A = -\frac{\cos \frac{1}{2}(b+c) \cos \frac{1}{2}(b-c)}{\sin^2 \frac{1}{2}b \sin^2 \frac{1}{2}c}. \qquad \text{(C.)}$$

7. D, E, F are the middle points of the sides of a triangle ABC. Prove

(i) AD, BE, CF are concurrent;

(ii) $2 \cos \frac{1}{2}a \cos AD = \cos b + \cos c$;

(iii) $4 \cos \frac{1}{2}b \cos \frac{1}{2}c \cos EF = 1 + \cos a + \cos b + \cos c$. (C.)

8. A spherical triangle varies subject to B and C remaining constant; prove that, for a small variation,

$$\delta A = \delta a \sin b \sin C = \delta b \sin A \tan c = \delta c \sin A \tan b.$$

9. Four points on a sphere are equally distant from one another; what is the angular distance between them?

10. Prove that if the spherical triangle, ABC, be such that the small circle on AB as diameter passes through C,

$$\cot A \cot B = \cos^2 \tfrac{1}{2}C.$$

In plane trigonometry this becomes $\tan A = \cot B$, which remains true for a large sphere if the square of the ratio of length of arc to the radius of the sphere is negligible.

CHAPTER XIV

MOMENTS OF INERTIA

189. *System of Particles.* A point, (x, y, z), to which a certain mass, m, is assigned, may be called a particle. Every system of particles possesses a certain symmetry, which is expressed by the existence of its centre of mass, and its momental ellipsoid at the centre of mass. This is a geometrical proposition which has applications to various calculations, not only those of dynamics. It is not necessary that the number, m, assigned to each point, should represent the mass of dynamics, it may have some other meaning, but it is assumed to be positive. The name centroid will be adopted here for centre of mass.

190. *Centroid.* Take coordinate axes in any position with regard to a given system of particles, P_1, P_2 ..., whose masses are m_1, m_2 ...; and let (x_1, y_1, z_1), (x_2, y_2, z_2) ... be their coordinates referred to these axes; and write M for Σm, the sum of the masses. Then the centroid, G, is defined as the point whose coordinates, $(\bar{x}, \bar{y}, \bar{z})$, are given by the equations

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M\bar{z} = \Sigma mz.$$

Here, and elsewhere, Σ denotes summation for all the particles of the system.

It will now be proved that the centroid has a unique position with reference to the system, independent of the choice of coordinate axes.

Take any plane

$$\lambda x + \mu y + \nu z = p,$$

and let q_1, q_2 ... be the algebraical distances of the particles

from this plane, and $\bar{q}$ the algebraical distance of the point G from it. Then

$$q_1 = \lambda x_1 + \mu y_1 + \nu z_1 - p,$$

$$q_2 = \lambda x_2 + \mu y_2 + \nu z_2 - p,$$

....

Therefore $\Sigma mq = \lambda \Sigma mx + \mu \Sigma my + \nu \Sigma mz - p \Sigma m$

$$= M(\lambda \bar{x} + \mu \bar{y} + \nu \bar{z} - p) = M\bar{q}.$$

This shows that, whatever set of planes at right angles to one another are taken as coordinate planes, the formulae giving the coordinates of the point G are the same.

191. *Moments of Inertia.* Let PN be the perpendicular drawn from a particle P to a given straight line. Then ΣmPN^2 is called the moment of inertia of the system of particles about this straight line.

Taking the coordinate axes in any position with regard to the system of particles, the moments of inertia of the system about the axes of x, y and z respectively are

$$\Sigma m(y^2 + z^2), \quad \Sigma m(z^2 + x^2), \quad \Sigma m(x^2 + y^2).$$

Also Σmyz, Σmzx and Σmxy are called the products of inertia of the system about these lines. Let

$$x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z',$$

where $(\bar{x}, \bar{y}, \bar{z})$ are the coordinates of the centroid, G. Then (x', y', z') are the coordinates of a particle (x, y, z) with regard to parallel axes through G; so that $\Sigma mx'$, $\Sigma my'$ and $\Sigma mz'$ are all zero. Therefore

$$\begin{aligned}
\Sigma m(y^2 + z^2) & \\
&= \Sigma m\{(\bar{y} + y')^2 + (\bar{z} + z')^2\} \\
&= M(\bar{y}^2 + \bar{z}^2) + \Sigma m(y'^2 + z'^2) + 2\bar{y}\Sigma my' + 2\bar{z}\Sigma mz' \\
&= M(\bar{y}^2 + \bar{z}^2) + \Sigma m(y'^2 + z'^2).
\end{aligned}$$

And we have corresponding results in terms of the other coordinates. This shows that the moment of inertia of

the given system, about any line, is equal to the moment of inertia of a particle of mass M, placed at G, about this line, plus the moment of inertia of the system about a parallel line through G.

Similarly

$$\Sigma myz = \Sigma m\,(\bar{y} + y')\,(\bar{z} + z') = M\bar{y}\bar{z} + \Sigma my'z',$$

with corresponding results with regard to the other coordinates. Therefore the products of inertia follow the rule that each of them is equal to the product of inertia of a particle of mass M placed at G, plus the product of inertia of the system about parallel lines through G.

Thus all moments and products of inertia can be derived from those about lines drawn through the centroid.

192. *Momental Ellipsoid.* Take any point, O, for origin. And through this point draw any straight line, with direction cosines (λ, μ, ν). Then the moment of inertia of the system about this line is, (§ 40),

$$\Sigma m\,\{x^2 + y^2 + z^2 - (\lambda x + \mu y + \nu z)^2\},$$

or $$\Sigma m\,\{(x^2 + y^2 + z^2)\,(\lambda^2 + \mu^2 + \nu^2) - (\lambda x + \mu y + \nu z)^2\},$$

which may be written

$$A\lambda^2 + B\mu^2 + C\nu^2 - 2D\mu\nu - 2E\nu\lambda - 2F\lambda\mu,$$

where

$$A = \Sigma m\,(y^2 + z^2), \quad B = \Sigma m\,(z^2 + x^2), \quad C = \Sigma m\,(x^2 + y^2),$$

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad F = \Sigma mxy.$$

Thus A, B, C are the moments of inertia, and D, E, F are the products of inertia, about the axes.

Draw the quadric surface

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = 1,$$

and let r be the length of the central radius of this surface in the direction (λ, μ, ν). Then the moment of inertia about this line is equal to $\dfrac{1}{r^2}$. Therefore, as there is a

positive moment of inertia about every radius, the surface must be an ellipsoid. And as the lengths of the diameters are specified by the moments of inertia of the system about them, this ellipsoid must be independent of the directions of the coordinate axes. It is called the momental ellipsoid at the point which is chosen for the origin, which may be any point*.

193. *Principal Axes.* Every ellipsoid has a set of three principal axes, (or more than one set if it is an ellipsoid of revolution). The principal axes of the momental ellipsoid, at any given point, are called the principal axes of the given system at that point. Let A', B', C' be the moments of inertia of the system about the principal axes at O. Then the equation of the momental ellipsoid at this point, referred to these axes, is

$$A'x^2 + B'y^2 + C'z^2 = 1.$$

This shows that the products of inertia about the principal axes are all zero. The moments of inertia about the principal axes at any given point are called the principal moments of inertia at that point.

Thus, at any given point, there is a momental ellipsoid with its centre at that point; also a set of principal axes or more than one set, and a set of principal moments of inertia. But the ellipsoid and moments of inertia at the centroid are the most important. From these all others can be derived.

* It should be noticed that the size of the momental ellipsoid, as defined here, depends on the choice of the units of length and mass. As in the case of other surfaces employed only for graphical representation, this need not be regarded as a defect if it happens to be convenient. It is assumed that some choice has been made, and it is not necessary to specify it. But it may sometimes be convenient to employ a momental ellipsoid of a definite size, in comparison with the given system, whatever units may be chosen. In that case it is specified by the equation

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = M\epsilon^4,$$

where M is the mass of the system, and ϵ is a length chosen arbitrarily. The ellipsoid is then independent of the choice of units of length and mass; and the moment of inertia about a radius is $\frac{M\epsilon^4}{r^2}$.

The moment of inertia of the system about a given straight line is usually written Mk^2, where M is the mass of the system, and k is called the radius of gyration of the system about this line.

The properties of an ellipsoid show that, if the moments of inertia of the system about lines through a given point are unequal the greatest is that about one of the principal axes, and the least is that about another of the principal axes. Also that if the moments of inertia about the principal axes are all equal, the moments of inertia about all lines through the same point are equal, the ellipsoid being a sphere. Also we have the invariants of § 103.

Any special symmetry which a system possesses can be used as a guide in the search for its principal axes. But there is only one general method for finding them, namely that of calculating a set of moments and products of inertia, and then finding the directions of the axes of the ellipsoid by means of the formulae of § 101.

194. *Ellipsoid of Gyration.* Let Ma^2, Mb^2, Mc^2 be the moments of inertia of the system about the principal axes at any point O, taken as origin. Then the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

referred to these axes, is called the ellipsoid of gyration at O. Let p be the length, and (λ, μ, ν) the direction cosines, of the perpendicular from O on a tangent plane of this surface. Then $p^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2$. Therefore Mp^2 is equal to the moment of inertia of the system about the line of this perpendicular. This ellipsoid has a definite size, independent of the choice of units.

If the ellipsoid of gyration at the centroid is known, it can be used for finding the principal axes and moments of inertia at any other point. The construction, (due to Clifford), by means of which this can be done, is given in the Appendix, § 235, note 5.

195. *Volumes and Plane Areas.* In the common case in dynamics of the given system being a homogeneous body, say of density ρ, we are practically dealing with volumes, the mass of a particle being an element of volume multiplied by ρ. Thus the moments of inertia about the axes are

$$\iiint\rho\,(y^2+z^2)\,dxdydz,\quad \iiint\rho\,(z^2+x^2)\,dxdydz,$$

$$\iiint\rho\,(x^2+y^2)\,dxdydz.$$

And we have corresponding expressions for the products of inertia.

By taking the moment of inertia of a plane area, assumed to have unit mass per unit area, we get what is called the moment of inertia of the area. This is useful in some calculations about areas. Total area takes the place of total mass.

Take the axes of x and y in the plane of the area. Then $dxdy$ is an element of area, and

$$A=\iint y^2dxdy,\quad B=\iint x^2dxdy,\quad C=\iint(x^2+y^2)\,dxdy,$$

$$D=0,\quad E=0,\quad F=\iint xydxdy.$$

Therefore $$C=A+B.$$

And the equation of the momental ellipsoid at the origin is

$$Ax^2+By^2+(A+B)\,z^2-2Fxy=1.$$

This shows that, at any point in the area, one of the principal axes is at right angles to the plane, and the moments of inertia about lines in the plane are the inverse squares of the radii of the ellipse

$$Ax^2+By^2-2Fxy=1.$$

Thus the axes of this ellipse are principal axes at the origin. If there is a straight line with regard to which the area is symmetrical, this line is obviously a principal axis at every point in it.

A body which may be treated as a plane area, of uniform mass per unit area, is called a lamina.

196. *Examples.*

1. Verify the following expressions for the moments of inertia about principal axes at the centroid:

i. Rectangular lamina, sides $2a$ and $2b$: $\frac{1}{3}Ma^2$, $\frac{1}{3}Mb^2$, $\frac{1}{3}M(a^2+b^2)$.

ii. Circular lamina, radius a: $\frac{1}{4}Ma^2$, $\frac{1}{4}Ma^2$, $\frac{1}{2}Ma^2$.

iii. Elliptic lamina, axes $2a$, $2b$: $\frac{1}{4}Ma^2$, $\frac{1}{4}Mb^2$, $\frac{1}{4}M(a^2+b^2)$.

iv. Rectangular body, edges $2a$, $2b$, $2c$: $\frac{1}{3}M(b^2+c^2)$, $\frac{1}{3}M(c^2+a^2)$, $\frac{1}{3}M(a^2+b^2)$.

v. Sphere, radius a: $\frac{2}{5}Ma^2$.

vi. Ellipsoid, semi-axes a, b, c: $\frac{1}{5}M(b^2+c^2)$, $\frac{1}{5}M(c^2+a^2)$, $\frac{1}{5}M(a^2+b^2)$.

2. Find the principal moments of inertia, at the vertex, of a solid right circular cone, of height h, and vertical angle 2α, and unit density. [Taking the vertex as origin, and the axis of cone for axis of y, the moment about the axis of x of a circular lamina, of radius r, at distance y from the origin, is $\pi r^2 dy\,(\frac{1}{4}r^2+y^2)$, or $\pi\tan^2\alpha\,(\frac{1}{4}\tan^2\alpha+1)\,y^4dy$; and integration gives the moment of inertia of the cone about the axis of x, namely $\frac{3}{5}M(\frac{1}{4}\tan^2\alpha+1)h^2$.]

3. A uniform rectangular plate whose sides are of lengths $2a$, $2b$, has a portion cut out in the form of a square whose centre is the centre of the rectangle, and whose mass is half the mass of the plate. Show that the axes of greatest and least moment of inertia at a corner of the rectangle make angles θ, $\frac{1}{2}\pi+\theta$ with a side, where

$$\tan 2\theta = \tfrac{8}{5}ab/(a^2-b^2). \qquad \text{(S 1.)}$$

(The plate is a lamina, and the moments of inertia are only about lines in the plane.)

4. A lamina, of area A, is symmetrical about an axis XY in its plane. An axis $X'Y'$ parallel to XY, and at distance h, lies in the plane of the lamina, and does not intersect it. I is the moment of inertia about $X'Y'$ of the solid, considered to be of unit density, formed by the rotation of A about $X'Y'$; i is the moment of inertia about XY of the lamina, considered to be of unit areal density. Prove that

$$I = 2\pi Ah^3 + 6\pi hi.$$

Hence, or otherwise, find the moment of inertia, about its axis, of an anchor ring formed by the rotation of a circle, of radius a, about an axis at distance c from its centre. (S 1.)

5. A uniform solid body is bounded by the planes $z=\pm 1$ and the hyperboloid $x^2+y^2-z^2=1$; calculate its moment of inertia about a generating line of the hyperboloid.

CHAPTER XV

STRAIN

197. Any change of the configuration of the points forming a figure, subject to the condition that contiguous points remain contiguous, is called a strain of the figure. The points must be conceived to be identifiable, like particles forming a body. We take an unstrained figure composed of points P, Q ......, and the corresponding strained and displaced figure, composed of the corresponding points P', Q' ......, and we propose to investigate the measurement of the strain, subject to certain conditions as to continuity of the relations between the two figures. It is assumed that the figure is in three dimensions, and therefore cannot by mere displacement be made to coincide with a figure which is its reflection (§ 9).

198. *Shifts.* Take coordinate axes, which will be assumed here to be attached to the unstrained figure; and let (x, y, z) be the coordinates of any point, P, of this figure. And let (X, Y, Z), or $(x+u, y+v, z+w)$, be the coordinates of the corresponding point, P', in the second figure, derived from the first figure by strain and displacement. Then u, v, w are called the shifts of the point P. They are the components, in the directions of the axes, of the step PP', (§ 10), which may be called the total shift of P.

The shifts, u, v, w, are to be regarded as functions of x, y and z. It will be assumed that they are continuous functions of these independent variables, with a single value at each point, and possessing differential coefficients of the first and higher orders, so far as we have occasion to use them. The nine partial differential coeffi-

cients of the first order, with regard to x, y and z, namely $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$......, will be denoted by u_x, u_y, u_z, v_x, v_y, v_z, w_x, w_y, w_z, and will be called the shift fluxions.

It will be noticed that two different sets of equations, giving u, v and w in terms of x, y and z, may represent the same strain, that is to say the same change of configuration of the points, the difference between them representing only a difference of bodily displacement of the figure.

199. *Stretch.* A stretch is a simple example of a strain. It is specified by a magnitude and a signless direction. To define a stretch, let k be its magnitude, and take the axis of x parallel to its direction. It is then specified by the equations

$$u = kx, \quad v = 0, \quad w = 0;$$

or
$$X = (1 + k)\,x, \quad Y = y, \quad Z = z.$$

A stretch for which $1 + k$ is positive will be called a stretch of the first type, and a stretch for which $1 + k$ is negative will be called a stretch of the second type. The value -1 for k is not admitted. The only effect of a stretch of the first type is an increase or decrease of the spacing of planes in the figure, at right angles to its direction, by a certain percentage. For example, if $k = 1$ the spacing is doubled, if $k = -\frac{1}{2}$ the spacing is halved. A stretch of the second type not only alters the spacing of planes in the figure at right angles to its direction, but also reverses the order of them. Therefore a stretch of the second type, though it may be used as a transformation of a geometrical figure, is not possible for a solid body. In connection with the strain of a body, a stretch means a stretch of the first type.

A stretch may be called a one dimensional strain; because it is completely specified by its effect on points in one straight line, parallel to the direction of the stretch.

The equations $u = kx$, $v = 0$, $w = 0$ show that the direction of a stretch is signless, (§ 5), because a reversal of the direction taken as the positive direction of the axis of x does not affect the strain which these equations represent.

200. *Homogeneous Strain.* This chapter will deal mainly with two mathematical theories: first the theory of homogeneous strain of any magnitude; and then the theory of small, or nascent, strain of a body, not required to be homogeneous.

A strain is said to be homogeneous if the shifts are linear functions of x, y, z, that is to say if they are given by equations of the form

$$u = (\lambda_1 - 1)\,x + \lambda_2 y + \lambda_3 z,$$
$$v = \mu_1 x + (\mu_2 - 1)\,y + \mu_3 z,$$
$$w = \nu_1 x + \nu_2 y + (\nu_3 - 1)\,z,$$

the origin being the point at which the shifts are zero. These equations may also be written

$$X = \lambda_1 x + \lambda_2 y + \lambda_3 z,$$
$$Y = \mu_1 x + \mu_2 y + \mu_3 z,$$
$$Z = \nu_1 x + \nu_2 y + \nu_3 z,$$

where (X, Y, Z) are the coordinates of the new position of the point (x, y, z). The coefficients λ_1, λ_2 ...... may be any given set of nine numbers, provided that

$$\lambda_1 (\mu_2 \nu_3 - \mu_3 \nu_2) + \lambda_2 (\mu_3 \nu_1 - \mu_1 \nu_3) + \lambda_3 (\mu_1 \nu_2 - \mu_2 \nu_1)$$

is not zero. This condition is needed in order to provide that each point in the second figure corresponds to a single point in the first figure. This will be seen by solving the equations so as to obtain x, y and z in terms of X, Y and Z. Let the result of this solution be

$$x = \lambda_1' X + \lambda_2' Y + \lambda_3' Z,$$
$$y = \mu_1' X + \mu_2' Y + \mu_3' Z,$$
$$z = \nu_1' X + \nu_2' Y + \nu_3' Z.$$

Then these equations can be solved so as to give the previous set, therefore

$$\lambda_1'(\mu_2'\nu_3' - \mu_3'\nu_2') + \lambda_2'(\mu_3'\nu_1' - \mu_1'\nu_3') + \lambda_3'(\mu_1'\nu_2' - \mu_2'\nu_1')$$

is not zero. Any one of these three sets of equations may be called the equations of transformation for the strain and displacement which they represent. And the displacement may be called angular displacement, because we have a point with zero shift, namely the origin.

The linearity of these equations shows, as in § 56, that a plane in one figure corresponds to a plane in the other figure, and a quadric surface in one figure to a quadric surface in the other. Points of intersection of two surfaces in one figure correspond to intersections of the corresponding surfaces in the other, and points at infinity in one figure to points at infinity in the other. Parallel planes in one figure correspond to parallel planes in the other, because they do not intersect; and a straight line corresponds to a straight line, because it is the intersection of two planes. Two parallel lines correspond to two parallel lines, because they are in one plane and do not intersect.

Let the first figure be divided into equal cubes, of length of edge a, by the coordinate planes and planes parallel to them. The equations of these planes are $x = na$, $y = na$, $z = na$, n being given in succession all integral values, positive or negative or zero. Therefore the equations of the corresponding planes in the second figure are

$$\lambda_1'X + \lambda_2'Y + \lambda_3'Z = na,$$

$$\mu_1'X + \mu_2'Y + \mu_3'Z = na,$$

$$\nu_1'X + \nu_2'Y + \nu_3'Z = na,$$

n being given in succession all integral values, positive or negative or zero. Thus they are three sets of parallel planes, each set being uniformly spaced, dividing the second figure into equal parallelepipeds. And as a may be as small as may be desired, the two sets of parallel

planes could be used, like corresponding sets of lines on squared paper, for the construction of either figure when the other is given, as accurately as might be desired. Thus they show that a strain specified by linear equations of transformation is homogeneous, in the ordinary sense of that term, the law of transformation of any figure being uniform throughout it.

Also the origin may be shifted to any point in the first figure, without changing the equations of transformation, provided that the second figure is at the same time shifted bodily, without rotation, so as to bring the point corresponding to the origin into coincidence with it.

The scheme of dividing the unstrained figure into cubes, which may be as small as we please, shows that the lengths of any set of parallel lines are all altered in the same ratio. But an independent proof of this proposition can be given as follows. Let (l, m, n) be the direction cosines of a line, of length r, drawn from a point (x_1, y_1, z_1) to a point (x_2, y_2, z_2) in the first figure; and let (L, M, N) be the direction cosines of the corresponding line, of length R, drawn from (X_1, Y_1, Z_1) to (X_2, Y_2, Z_2), in the second figure, then

$$LR = X_2 - X_1 = \lambda_1 (x_2 - x_1) + \lambda_2 (y_2 - y_1) + \lambda_3 (z_2 - z_1)$$
$$= r (\lambda_1 l + \lambda_2 m + \lambda_3 n).$$

Similarly $$MR = r (\mu_1 l + \mu_2 m + \mu_3 n),$$

and $$NR = r (\nu_1 l + \nu_2 m + \nu_3 n).$$

Therefore

$$\frac{R^2}{r^2} = (\lambda_1 l + \lambda_2 m + \lambda_3 n)^2 + (\mu_1 l + \mu_2 m + \mu_3 n)^2 + (\nu_1 l + \nu_2 m + \nu_3 n)^2,$$

which shows that the ratio $R : r$ depends only on l, m, n.

201. *The Inverse Strain Ellipsoid.* This result can be exhibited by means of the quadric surface

$$(\lambda_1 x + \lambda_2 y + \lambda_3 z)^2 + (\mu_1 x + \mu_2 y + \mu_3 z)^2 + (\nu_1 x + \nu_2 y + \nu_3 z)^2 = 1.$$

This is the surface in the first figure which corresponds to the sphere $X^2 + Y^2 + Z^2 = 1$ in the second figure. It cannot extend to infinity, therefore it is an ellipsoid. It will be called the inverse strain ellipsoid*.

Each central radius of the ellipsoid is converted by the strain into a straight line of unit length. Therefore, if d is the distance between two points, P and Q, in the unstrained figure, and r is the length of the central radius of the ellipsoid parallel to PQ; then the distance between the corresponding points, P' and Q', in the strained figure is d/r. Accordingly, if all the points of the unstrained figure are given, the ellipsoid specifies all the distances between the corresponding points of the strained figure. Now we know that this information gives two alternative configurations for the strained figure, each of which is the reflection of the other (§ 9). Therefore, subject to this ambiguity, the ellipsoid specifies the strain.

If the strain is given the ellipsoid is specified, because the lengths of all its diameters are known. Thus it is not in any way dependent on the directions of the coordinate axes, or on the displacement of the figure as a whole. Also any ellipsoid, drawn in the unstrained figure, specifies two alternative homogeneous strains.

The equation of the ellipsoid will be written

$$(1 + 2\epsilon_1)\, x^2 + (1 + 2\epsilon_2)\, y^2 + (1 + 2\epsilon_3)\, z^2 + 2s_1 yz + 2s_2 zx + 2s_3 xy = 1,$$

where

$$\begin{aligned}
2\epsilon_1 &= \lambda_1^2 + \mu_1^2 + \nu_1^2 - 1, \\
2\epsilon_2 &= \lambda_2^2 + \mu_2^2 + \nu_2^2 - 1, \\
2\epsilon_3 &= \lambda_3^2 + \mu_3^2 + \nu_3^2 - 1, \\
s_1 &= \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3, \\
s_2 &= \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1, \\
s_3 &= \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2.
\end{aligned}$$

* It cannot be called the strain ellipsoid, because this name has been appropriated, long ago, to a different surface, see § 213. And "inverse" expresses the relation between the two surfaces better than the term "reciprocal," which is sometimes used.

The six numbers ϵ_1, ϵ_2, ϵ_3, s_1, s_2, s_3, which specify the ellipsoid, are called the components of the strain. If they are zero, there is either no strain, or else the second figure is the reflection of the first. This agrees with the formulae for change of the coordinate axes; for it will be seen, (§ 50), that the six components of strain being all zero is the condition that the equations of transformation are those which give merely a change of the directions of the coordinate axes with regard to the figure. The case of no strain corresponds to a rotational change of the directions of the axes; and the case in which the second figure is the reflection of the first corresponds to an irrotational change.

For a given strain, referred to given coordinate axes, there is an endless variety of possible sets of equations of transformation, differing in respect of the angular displacement which they specify. The simplest forms of them will now be sought.

202. *Principal Axes.* For a given strain, a change of the directions of the coordinate axes does not change the ellipsoid, but only the equation of it. Taking the principal axes of the surface for coordinate axes, the equation of the surface may be written

$$A^2x^2 + B^2y^2 + C^2z^2 = 1,$$

where A, B and C are positive numbers. These axes are called the principal axes of the strain, and are regarded as a set of lines in the unstrained figure with signless directions. In general they are a unique set of lines. They are not unique when the ellipsoid, and therefore also the strain, is symmetrical with regard to one of them; in that case any set of principal axes of the ellipsoid may be taken as the principal axes of the strain.

203. *Expression of the strain in terms of Stretches.* This form of the equation of the ellipsoid shows that,

referred to the principal axes, any one of the sets of equations

$$X = \pm Ax, \quad Y = \pm By, \quad Z = \pm Cz,$$

any combination of signs being chosen, is a possible set of equations of transformation for a strain specified by this ellipsoid. But a given ellipsoid specifies two, and only two, alternative strains. Therefore any two of these sets of equations, chosen so that they give different strains, will serve to represent the two strains specified by the ellipsoid. Let us choose the two sets of equations

$$X = Ax, \quad Y = By, \quad Z = Cz,$$

and

$$X = -Ax, \quad Y = By, \quad Z = Cz.$$

They represent different strains, for each set gives a second figure which is the reflection of that given by the other set (§ 9). Also neither of these strains disturbs the directions of lines in the first figure which are the principal axes, or parallel to them.

The equations

$$X = Ax, \quad Y = By, \quad Z = Cz,$$

which may be written

$$u = (A - 1)\, x, \quad v = (B - 1)\, y, \quad w = (C - 1)\, z,$$

represent a combination of three stretches of the first type, (§ 199), of magnitudes $A - 1$, $B - 1$, $C - 1$, in the directions of the principal axes of the strain. Their effect is the same whether they are imposed simultaneously, or singly in any order. Dividing the unstrained figure, as before, into equal cubes, they convert these cubes into rectangular parallelepipeds. This is the most general homogeneous strain that is possible for a body. It will be called a strain of the first type.

The equations

$$X = -Ax, \quad Y = By, \quad Z = Cz$$

similarly represent a combination of three stretches, of magnitudes $-A - 1$, $B - 1$, $C - 1$; but one of them is a stretch of the second type, reversing the order of planes

at right angles to one of the principal axes. This is the alternative strain which a given ellipsoid specifies. It is not a possible strain for a body, and will be called a strain of the second type.

The three stretches, in the directions of the principal axes, of which any given homogeneous strain is composed, are called its principal stretches. In the case of the strain of a body, they are a unique set of three stretches.

In actual calculations, the use of this simple scheme of principal axes and principal stretches, does not occur so often as might be expected; partly because finding their directions is often troublesome; and partly because, when a strain is not homogeneous, their directions vary from point to point of the figure or body; and partly because the directions of the coordinate axes which it is convenient to use are often settled by some quite different consideration.

204. *Pure Strain and Angular Displacement.* The relation of the ellipsoid,

$$A^2x^2 + B^2y^2 + C^2z^2 = 1,$$

to the corresponding sphere,

$$X^2 + Y^2 + Z^2 = 1,$$

has been discussed in a previous chapter. It has been proved there, (§ 74), that conjugate diameters of the ellipsoid correspond to sets of diameters of the sphere at right angles to one another. And as principal axes of the ellipsoid are the only conjugate diameters which are at right angles to one another, principal axes of the strain are the set, or sets, of lines at right angles, in the first figure, which remain at right angles to one another after strain.

The discovery of this property of the principal axes, namely that they are lines at right angles in the unstrained figure which correspond to lines at right angles in the strained figure, enables us to give a definite meaning to

angular displacement of a figure, as distinguished from its strain; for we can define it as being the angular displacement of the principal axes. When angular displacement is referred to, without qualification, it is understood to have this meaning. A strain applied so that the principal axes retain their signless directions unchanged is called a pure strain. Thus any pure strain can be represented either by

$$X = Ax, \quad Y = By, \quad Z = Cz,$$

or by $$X = -Ax, \quad Y = By, \quad Z = Cz,$$

as equations of transformation, if the coordinate axes are properly chosen. And angular displacement will be understood to mean that which has to be applied, to supplement a pure strain, in order to complete the requirements of any given set of equations of transformation. Here A, B and C are positive numbers.

205. *Conditions for Pure Strain.* It will now be proved that, if the equations of transformation

$$X = \lambda_1 x + \lambda_2 y + \lambda_3 z,$$

$$Y = \mu_1 x + \mu_2 y + \mu_3 z,$$

$$Z = \nu_1 x + \nu_2 y + \nu_3 z,$$

represent a pure strain, $\mu_3 - \nu_2$, $\nu_1 - \lambda_3$ and $\lambda_2 - \mu_1$ are all zero. Take any homogeneous strain, and coordinate axes in any given directions, and let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the direction cosines of the principal axes of the strain, and $A - 1$, $B - 1$, $C - 1$ the magnitudes of the principal stretches, where A, B and C may now be either positive or negative. Then the equations of transformation which represent this strain, applied so as to be pure, can be written

$$\alpha_1 X + \beta_1 Y + \gamma_1 Z = A(\alpha_1 x + \beta_1 y + \gamma_1 z),$$

$$\alpha_2 X + \beta_2 Y + \gamma_2 Z = B(\alpha_2 x + \beta_2 y + \gamma_2 z),$$

$$\alpha_3 X + \beta_3 Y + \gamma_3 Z = C(\alpha_3 x + \beta_3 y + \gamma_3 z);$$

for each of these equations expresses the effect of one of the three independent stretches, applied so as to be pure. Now these equations can be put into the standard form,

$$X = \lambda_1 x + \lambda_2 y + \lambda_3 z \text{......}$$

by solving them for X, Y and Z. Multiplying the first equation by α_1, the second by α_2, and the third by α_3, and adding, and taking account of the properties of the direction cosines of sets of lines at right angles, we get the first of the standard equations, namely

$$X = (A\alpha_1^2 + B\alpha_2^2 + C\alpha_3^2)\,x + (A\alpha_1\beta_1 + B\alpha_2\beta_2 + C\alpha_3\beta_3)\,y \\ + (A\gamma_1\alpha_1 + B\gamma_2\alpha_2 + C\gamma_3\alpha_3)\,z.$$

And the symmetry of the coefficients of y and z shows that the coefficient of y, in this equation for X, is equal to the coefficient of x in the corresponding equation for Y. That is to say $\lambda_2 = \mu_1$. Similarly $\mu_3 = \nu_2$, and $\nu_1 = \lambda_3$.

That the converse of this proposition is also true can be proved as follows. Take a set of equations of transformation for which $\mu_3 = \nu_2$, $\nu_1 = \lambda_3$ and $\lambda_2 = \mu_1$, then these equations may be written

$$X = \lambda_1 x + \lambda_2 y + \nu_1 z,$$

$$Y = \lambda_2 x + \mu_2 y + \mu_3 z,$$

$$Z = \nu_1 x + \mu_3 y + \nu_3 z.$$

Now these equations, involving only six coefficients, have an obvious relation to the surface consisting of the pair of quadrics

$$\lambda_1 x^2 + \mu_2 y^2 + \nu_3 z^2 + 2\mu_3 yz + 2\nu_1 zx + 2\lambda_2 xy = \pm k.$$

This surface is either a single ellipsoid, or else a pair of conjugate hyperboloids; and k can be chosen so as to make it pass through any given point, P, of the first figure. And the equations of transformation show that the direction, OP', of the radius from the origin to the corresponding point, P', in the second figure, is perpen-

dicular to the tangent plane of the surface at P. Now we know that there are three directions from the origin, at right angles to one another, along which a radius of the surface is also a normal, namely the directions of the principal axes of the surface. Therefore these are the principal axes of the strain, and have no angular displacement. That is to say the equations of transformation represent a pure strain.

206. *Homogeneous strain of a Body.* Let us now pay attention only to strains of the first type, that is to say the strains whose principal stretches are each of them greater than -1. These are the homogeneous strains which are possible for a body. Accordingly this restriction may be marked by the use of the term "body" instead of "figure." And A, B and C are restricted to being positive.

Let the principal stretches, $A-1$, $B-1$, $C-1$, of a homogeneous strain, be applied to a body in succession. It is obvious that the first stretch converts any volume, V, of the body into a volume AV; similarly the second converts this into a volume ABV; and the third converts this into a volume $ABCV$. Thus the ratio of the increase of the volume to the original volume is $ABC-1$. It is called the dilatation due to the strain. This comparison of volumes implies that we are dealing only with strains of the first type.

A stretch has been called a one dimensional strain. A strain which is compounded of two principal stretches, the third principal stretch being zero, may similarly be called a two dimensional strain, because it can be fully represented by the change of configuration of points in one plane, namely a plane parallel to the directions of the two stretches. The strain may be referred to as being in this plane. All lines at right angles to this plane remain at right angles to it.

A two dimensional homogeneous strain for which the

dilatation is zero is called a slide*. Thus a slide is a strain which consists of two stretches, $A-1$ and $B-1$, in directions at right angles to one another, with the condition $AB-1=0$. Accordingly a slide is specified by two signless directions at right angles to one another, and one magnitude.

The stretch and the slide are types of strain which are suggested as elementary types by the general form of a set of equations of transformation. For we have seen that each of the three sets of equations,

$$u=(\lambda_1-1)\,x, \quad v=0, \quad w=0,$$
$$u=0, \quad v=(\mu_2-1)\,y, \quad w=0,$$
$$u=0, \quad v=0, \quad w=(\nu_3-1)\,z,$$

specifies a stretch. And it will now be shown that the strain specified by each of the six sets of equations

$$u=0, \quad v=0, \quad w=\nu_2 y,$$
$$u=0, \quad v=\mu_3 z, \quad w=0,$$
$$u=\lambda_3 z, \quad v=0, \quad w=0,$$
$$u=0, \quad v=0, \quad w=\nu_1 x,$$
$$u=0, \quad v=\mu_1 x, \quad w=0,$$
$$u=\lambda_2 y, \quad v=0, \quad w=0,$$

is a slide.

Consider the effect of the first of these sets of equations taken alone, namely

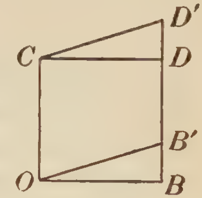
$$u=0, \quad v=0, \quad w=\nu_2 y.$$

It is two dimensional, so it can be represented by a diagram in one plane, namely the plane of yz. Draw a square $OBDC$ in the unstrained body, OB and OC being

* In adopting the terms "shift," "stretch" and "slide" in preference to the alternatives "displacement," "extension" and "shear," I follow Todhunter and Pearson's *History of Elasticity*. "Slide" has the advantage of corresponding best to the French term "glissement." And it is convenient to reserve the word "shear" for use in connection with stress, so that shearing stress may be called simply shear.

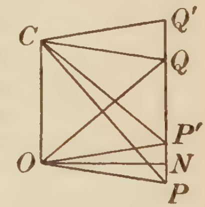
the axes of y and z. This is converted into the parallelogram $OB'D'C$, such that

$$DD' = BB' = \nu_2 OB.$$



And the area of the parallelogram is equal to that of the square, therefore the dilatation is zero, therefore the strain is a slide. The magnitude of the slide is defined as ν_2. Each of the cubes into which the body is conceived to be divided, by the coordinate planes and planes parallel to them, is altered in this way.

To show the principal axes of this strain, and the angular displacement specified by these equations, a more symmetrical diagram must be drawn. Divide the body, in the plane of yz, into equal rhombuses, such as $OPQC$, instead of squares. Here OC is the axis of z, and the rhombus, $OPQC$, is drawn so that the shift, PP', of the point P, is bisected at N by the axis of y. Accordingly ν_2 is the ratio of PP' to ON.



The rhombus $OPQC$ is converted into the rhombus $OP'Q'C$, differing from the former in having the acute and obtuse angles interchanged. The diagonals, CP, OQ, of the first rhombus, are at right angles, and they correspond to the diagonals CP', OQ' of the second rhombus, which are also at right angles. Therefore CP, OQ and a line parallel to the axis of x are the principal axes of the strain. And the angular displacement specified by the equations is the angular displacement of the principal axes, namely the angle PCP'. This is equal to half the angle POP', because P, P' and C are points on a circle with centre O. By placing the two rhombuses with their centres and diagonals coincident it will be seen that the principal stretches are $\dfrac{OQ' - OQ}{OQ}$ and $\dfrac{CP' - CP}{CP}$, the third principal stretch being zero.

The diagram is drawn for the case in which ν_2 is positive. A diagram for the case in which ν_2 is negative is obtained by interchanging P and P', and Q and Q'.

Let the angle of the rhombus at P be 2θ. It is an acute angle if ν_2 is positive, and an obtuse angle if ν_2 is negative. In either case

$$\nu_2 = 2 \cot 2\theta = \cot \theta - \tan \theta.$$

Also the ratio of the diagonals is $\tan \theta$; therefore the principal stretches are $\cot \theta - 1$ and $\tan \theta - 1$. Thus ν_2 is equal to the difference of these two stretches. And the angular displacement is an angle whose tangent is either $\frac{1}{2}\nu_2$ or $-\frac{1}{2}\nu_2$. The angular displacement in the plane of yz will be reckoned positive when it makes a right handed screw with the direction of the axis of x. This is the usual convention in physics. Therefore, taking the axes to be drawn according to the same convention, the axis of x making a right handed screw with rotation from the axis of y towards the axis of z, the angular displacement for the equations which we are considering is the angle whose tangent is $\frac{1}{2}\nu_2$.

This shows that for a small strain, when 2θ is nearly a right angle, the angular displacement tends to the value $\frac{1}{2}\nu_2$, and the principal stretches tend to the values $\pm \frac{1}{2}\nu_2$.

The second set of equations, $u = 0$, $v = \mu_3 z$, $w = 0$, gives a slide and angular displacement in the same plane of yz. But it gives a slightly different result in one respect, namely that the angular displacement, according to the convention adopted, is the angle whose tangent is $-\frac{1}{2}\mu_3$, and thus becomes $-\frac{1}{2}\mu_3$ for a small strain.

The third and fourth sets of equations give corresponding results in the plane of zx, and the fifth and sixth sets give corresponding results in the plane of xy.

These stretches and slides, represented by the nine coefficients in the equations of transformation, taken separately, are important because they can be super-

posed geometrically in the theory of small strain, as will be seen below. In the theory that we are at present concerned with, namely homogeneous strain of any magnitude, the strain of a cube in the body, with edges parallel to the axes, can be calculated as follows.

207. Let OA, OB, OC be the edges of this cube, and OA', OB', OC' the corresponding edges of the parallelepiped into which it is converted by the strain; and let the equations of transformation be

$$X = \lambda_1 x + \lambda_2 y + \lambda_3 z,$$

$$Y = \mu_1 x + \mu_2 y + \mu_3 z,$$

$$Z = \nu_1 x + \nu_2 y + \nu_3 z.$$

Then if a is the length of an edge of the cube, the coordinates of A' are $(\lambda_1 a,\ \mu_1 a,\ \nu_1 a)$; therefore, (§ 201),

$$OA'^2 = (1 + 2\epsilon_1)\, a^2,$$

and the elongation of OA, namely $\dfrac{OA' - OA}{OA}$, is $\sqrt{(1 + 2\epsilon_1)} - 1$. And we have corresponding expressions for the elongations of OB and OC. Also the direction cosines of OA', OB', OC' are proportional respectively to

$$(\lambda_1, \mu_1, \nu_1), \quad (\lambda_2, \mu_2, \nu_2), \quad (\lambda_3, \mu_3, \nu_3).$$

Therefore the cosine of the angle between OB' and OC' is

$$\frac{\lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3}{\sqrt{(1 + 2\epsilon_2)}\,\sqrt{(1 + 2\epsilon_3)}}, \quad \text{or} \quad \frac{s_1}{\sqrt{(1 + 2\epsilon_2)}\,\sqrt{(1 + 2\epsilon_3)}}.$$

And we have corresponding expressions for the cosines of the angles between OC' and OA', and between OA' and OB'. Thus the three elongations, and the cosines of the three angles, are all expressed in terms of the six components of strain, namely ϵ_1, ϵ_2, ϵ_3, s_1, s_2, s_3.

When the components of strain are small fractions, whose squares and products are negligible, the elongations become ϵ_1, ϵ_2, ϵ_3, and the cosines become s_1, s_2, s_3.

208. *General theory of Strain of a Body.* Let us now consider the general theory of the strain of a body, not restricted to being homogeneous. The shifts, u, v, w, are not now required to be linear functions of the coordinates, x, y, z, of a particle of the unstrained body. But they are assumed, (§ 198), to be continuous functions of x, y, z, possessing differential coefficients. And the nine differential coefficients of the first order, of u, v and w, are called the shift fluxions.

At any particular point of the body there is a certain strain and angular displacement. To explain what this means, let us once more conceive the unstrained body to be divided into equal cubes by the coordinate planes and planes parallel to them. If the strain is not homogeneous, the corresponding surfaces in the strained body are not sets of parallel planes uniformly spaced, dividing the body into parallelepipeds all alike. But they are surfaces which have tangent planes at all points, and which divide the strained body into portions with which it can be built up, as with bricks. And in the limiting case, in which the cubes are conceived to be infinitesimal, each portion approaches the form of a parallelepiped. And each of these parallelepipeds specifies, by its shape and size and orientation, the strain and angular displacement at a point, in the same way as if the parallelepipeds were all alike and the strain homogeneous. And the fact that the bricks have to fit together shows that the variation of angular displacement, from point to point, depends on the variation of the strain. And the variation of both strain and angular displacement must be compatible with their derivation from a possible set of shifts, u, v, w.

Taking coordinate axes, as before, attached to the unstrained body, let P, (x, y, z), be any point of this body, and Q, $(x + \delta x, y + \delta y, z + \delta z)$, an adjacent point of it, whose distance from P is to become infinitesimal; and let P', Q' be the corresponding points of the strained

body. Then the coordinates of P' are $(x+u, y+v, z+w)$, and the coordinates of Q' are

$$(x+\delta x+u+\delta u,\quad y+\delta y+v+\delta v,\quad z+\delta z+w+\delta w).$$

Now let the origin be moved to P, the strained body being at the same time conceived to be moved bodily, without rotation, so that P' may coincide with P, (§ 200). Then $(\delta x, \delta y, \delta z)$ are the new coordinates of Q, and $(\delta x+\delta u, \delta y+\delta v, \delta z+\delta w)$ are the new coordinates of Q'. And in the limiting case in which PQ becomes infinitesimal

$$\delta u = u_x\delta x + u_y\delta y + u_z\delta z,$$

$$\delta v = v_x\delta x + v_y\delta y + v_z\delta z,$$

$$\delta w = w_x\delta x + w_y\delta y + w_z\delta z,$$

where u_x, u_y ...... are the values of the nine shift fluxions at the point P. Thus in the immediate neighbourhood of each point of the body, there is a homogeneous strain, combined with angular displacement, specified by equations of transformation of the type

$$u = (\lambda_1 - 1)\,x + \lambda_2 y + \lambda_3 z,$$

$$v = \mu_1 x + (\mu_2 - 1)\,y + \mu_3 z,$$

$$w = \nu_1 x + \nu_2 y + (\nu_3 - 1)\,z,$$

in which the nine coefficients are the values of the nine shift fluxions at this point.

This shows that all the results which have been calculated for homogeneous strain of a body are also correct, when the strain is not homogeneous, for the strain at any given point. In order to use them we have only to substitute the nine shift fluxions, u_x, u_y, u_z, v_x, v_y, v_z, w_x, w_y, w_z, for the nine coefficients $\lambda_1 - 1$, λ_2, λ_3, μ_1, $\mu_2 - 1$, μ_3, ν_1, ν_2, $\nu_3 - 1$ respectively. Accordingly the strain at P is specified by the six components, ϵ_1, ϵ_2, ϵ_3, s_1, s_2, s_3, at that point, which must now be written

$$u_x + \tfrac{1}{2}(u_x^2 + v_x^2 + w_x^2),$$

$$v_y + \tfrac{1}{2}(u_y^2 + v_y^2 + w_y^2),$$

$$w_z + \tfrac{1}{2}(u_z^2 + v_z^2 + w_z^2),$$

$$w_y + v_z + u_y u_z + v_y v_z + w_y w_z,$$

$$u_z + w_x + u_z u_x + v_z v_x + w_z w_x,$$

$$v_x + u_y + u_x u_y + v_x v_y + w_x w_y.$$

Each point of the body has also its own inverse strain ellipsoid, and its own principal axes and principal stretches, and its own angular displacement relative to the unstrained body.

209. *The Theory of Elasticity.* In all the ordinary problems of the theory of elasticity of a solid body, the components of strain are very small fractions, whose squares and products are negligible in comparison with their first powers. And in most of these problems the angular displacements are also very small fractions, though some of them may not be quite so small as the components of strain.

Accordingly, the standard theory of elasticity adopts, as one of its data, what is called "the theory of small strain." This is a theory in which all the nine shift fluxions, not merely the components of strain, are treated as fractions small enough for their squares and products to be negligible. Thus the theory of small strain is an exact mathematical theory of infinitesimal shift fluxions, giving what Newton might have called nascent strains and angular displacements.

210. *Small Strain.* Thus in the theory of small strain the components of strain are

$$u_x, \quad v_y, \quad w_z, \quad w_y + v_z, \quad u_z + w_x, \quad v_x + u_y.$$

And for the sake of clearness it will be convenient to

use a special notation for these forms of the components of strain, namely

$$e_1, \quad e_2, \quad e_3, \quad \phi_1, \quad \phi_2, \quad \phi_3.$$

Also we can now simply add together, or superpose, the separate effects of the nine shift fluxions, taken one at a time as in § 206, because they do not produce any finite change in the form of the body; and the order in which they are considered is a matter of no consequence.

The three shift fluxions u_x, v_y and w_z give three pure stretches, of these magnitudes, in the directions of the axes. The shift fluxion w_y gives a slide of this magnitude in the plane of yz, which is expressed as a pure strain by two stretches, $\pm \frac{1}{2} w_y$, in directions bisecting the angles between the axes of y and z; and this pure strain is accompanied by an angular displacement, equal to $\frac{1}{2} w_y$, about a line in the positive direction of the axis of x. The shift fluxion v_z gives, in the same way, a slide in the same plane consisting of stretches $\pm \frac{1}{2} v_z$ in the same directions, accompanied by an angular displacement, equal to $-\frac{1}{2} v_z$, about the same line. Therefore, adding these stretches, w_y and v_z taken together give a single slide, $w_y + v_z$, in the plane of yz, consisting of stretches $\pm \frac{1}{2}(w_y + v_z)$ in directions bisecting the angles between the axes of y and z; and this pure strain is accompanied by an angular displacement, $\frac{1}{2}(w_y - v_z)$, about a line in the positive direction of the axis of x. In the same way, u_z and w_x combined give a pure slide, $u_z + w_x$, in the plane of zx, consisting of stretches $\pm \frac{1}{2}(u_z + w_x)$, accompanied by an angular displacement, $\frac{1}{2}(u_z - w_x)$, about a line in the positive direction of the axis of y. And v_x and u_y give a pure slide, $v_x + u_y$, in the plane of xy, consisting of stretches $\pm \frac{1}{2}(v_x + u_y)$, accompanied by an angular displacement, $\frac{1}{2}(v_x - u_y)$, about a line in the positive direction of the axis of z. The slides do not contribute any dilatation, therefore the total dilatation at any point is

$$(1 + u_x)(1 + v_y)(1 + w_z) - 1,$$

that is to say $u_x + v_y + w_z$, or $e_1 + e_2 + e_3$.

The sign of each angular displacement here follows the convention adopted in § 206.

Infinitesimal angular displacements are compounded in the same way as angular velocities, namely by the parallelogram law (§ 220). Thus $\frac{1}{2}(w_y - v_z)$, $\frac{1}{2}(u_z - w_x)$ and $\frac{1}{2}(v_x - u_y)$, which will be denoted by α, β, γ, are the components, according to this law, of the angular displacement at a point (x, y, z) of the body. Differentiation of these components shows that they satisfy the equation

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0;$$

that is to say, the angular displacement has a solenoidal vector distribution in the body, see § 224.

Thus we have seen that, in the case of infinitesimal, or nascent, strain and angular displacement, if the unstrained body is conceived to be divided into elementary cubes, by the coordinate planes, and planes parallel to them, the cube at any given point of the body is converted into a parallelepiped by stretches e_1, e_2, e_3 in the directions of the axes, and changes of the angles between its faces equal to ϕ_1, ϕ_2, ϕ_3, and receives an angular displacement whose components in the directions of the axes are α, β, γ. Now any set of infinitesimal values given to these nine quantities will specify the form and orientation of a single parallelepiped; but if they vary from point to point they must do so in such a way that the parallelepipeds will fit together. Accordingly it is necessary to take account of the relations between them, namely that they must be compatible with their derivation from a set of shifts, u, v, w, which are functions of x, y and z. If the shifts are known, and the strains and angular displacements are derived from them, compatibility is secured. But in problems in the theory of elasticity, the data are relations between strains, and the shifts are not known; so the conditions of compatibility become important, except in certain cases in which a

problem has some feature of symmetry which sufficiently indicates compatibility without special investigation.

The conditions of compatibility, or equivalent data derived from symmetry, enable us in many cases to dispense with actual consideration of the shifts, and to deal only with strains.

211. *Conditions of Compatibility.* The conditions of compatibility which must be satisfied by the six components of strain can be found as follows. We have the equations

$$e_1 = u_x, \quad e_2 = v_y, \quad e_3 = w_z,$$

$$\phi_1 = w_y + v_z, \quad \phi_2 = u_z + w_x, \quad \phi_3 = v_x + u_y,$$

$$2\alpha = w_y - v_z, \quad 2\beta = u_z - w_x, \quad 2\gamma = v_x - u_y.$$

And the fact that u_x, u_y, u_z are differential coefficients of a function of x, y and z is expressed by the three equations

$$\frac{\partial}{\partial y}(u_z) = \frac{\partial}{\partial z}(u_y), \quad \frac{\partial}{\partial z}(u_x) = \frac{\partial}{\partial x}(u_z), \quad \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial y}(u_x);$$

or

$$\frac{\partial}{\partial y}(\tfrac{1}{2}\phi_2 + \beta) = \frac{\partial}{\partial z}(\tfrac{1}{2}\phi_3 - \gamma), \quad \frac{\partial e_1}{\partial z} = \frac{\partial}{\partial x}(\tfrac{1}{2}\phi_2 + \beta),$$

$$\frac{\partial}{\partial x}(\tfrac{1}{2}\phi_3 - \gamma) = \frac{\partial e_1}{\partial y}.$$

These three equations, combined with the equation

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0,$$

give

$$\frac{\partial \alpha}{\partial x} = \frac{1}{2}\frac{\partial \phi_2}{\partial y} - \frac{1}{2}\frac{\partial \phi_3}{\partial z}, \quad \frac{\partial \beta}{\partial x} = \frac{\partial e_1}{\partial z} - \frac{1}{2}\frac{\partial \phi_2}{\partial x}, \quad \frac{\partial \gamma}{\partial x} = \frac{1}{2}\frac{\partial \phi_3}{\partial x} - \frac{\partial e_1}{\partial y};$$

and we have two similar sets of equations derived in the same way from v and w, nine equations altogether.

Now the equations which involve α are

$$\frac{\partial \alpha}{\partial x} = \frac{1}{2}\frac{\partial \phi_2}{\partial y} - \frac{1}{2}\frac{\partial \phi_3}{\partial z}, \quad \frac{\partial \alpha}{\partial y} = \frac{1}{2}\frac{\partial \phi_1}{\partial y} - \frac{\partial e_2}{\partial z}, \quad \frac{\partial \alpha}{\partial z} = -\frac{1}{2}\frac{\partial \phi_1}{\partial z} + \frac{\partial e_3}{\partial y};$$

and the fact that α is a function of x, y and z is expressed by the equations

$$\frac{\partial}{\partial y}\left(\frac{\partial \alpha}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \alpha}{\partial y}\right), \quad \frac{\partial}{\partial z}\left(\frac{\partial \alpha}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial z}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \alpha}{\partial x}\right),$$

that is to say

$$\frac{\partial^2 \phi_1}{\partial y\,\partial z} = \frac{\partial^2 e_2}{\partial z^2} + \frac{\partial^2 e_3}{\partial y^2}, \quad 2\frac{\partial^2 e_3}{\partial x\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} - \frac{\partial \phi_3}{\partial z}\right),$$

$$2\frac{\partial^2 e_2}{\partial z\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}\right).$$

Two corresponding sets of three equations are derived from β and γ being functions of x, y and z. But it is clear that three of the nine equations thus obtained occur twice, so that we get only six independent equations, namely

$$\frac{\partial^2 \phi_1}{\partial y\,\partial z} = \frac{\partial^2 e_2}{\partial z^2} + \frac{\partial^2 e_3}{\partial y^2}, \quad \frac{\partial^2 \phi_2}{\partial z\,\partial x} = \frac{\partial^2 e_3}{\partial x^2} + \frac{\partial^2 e_1}{\partial z^2}, \quad \frac{\partial^2 \phi_3}{\partial x\,\partial y} = \frac{\partial^2 e_1}{\partial y^2} + \frac{\partial^2 e_2}{\partial x^2},$$

$$2\frac{\partial^2 e_1}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}\right),$$

$$2\frac{\partial^2 e_2}{\partial z\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}\right),$$

$$2\frac{\partial^2 e_3}{\partial x\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} - \frac{\partial \phi_3}{\partial z}\right).$$

These six equations are the conditions of compatibility which must be satisfied by the components of strain, in order that they may be consistent with the existence of possible shifts, u, v, w. They were first given by Saint-Venant, in 1860, and were reproduced in his edition of Navier in 1864. It is a simple matter to verify them, and thus show that they are necessary. The object of a more elaborate investigation is to show that they are also sufficient to secure compatibility.

212. *Practical applications.* Practically the theory of small strain is usually applicable to problems in the theory

of elasticity in which the strains are small, though not infinitesimal, because usually the requirement that the angular displacements must be small can also be satisfied. But there are exceptions to this. For there are cases in which very small strain may necessarily imply a considerable angular displacement at some distance from the origin. If the strained body is built up with elementary parallelepipeds, each of which differs very little from a cube, the necessary cumulative effect in the form of angular displacement may be large.

The practical cases in which this happens are those in which the body has one dimension very small compared with other dimensions. In the bending of a thin rod, considerable angular displacement, relative to the unstrained body, may occur with very small strain. In such a case, in order to use the standard theory of elasticity, a special contrivance must be employed, which practically amounts to changing the origin and the axes as we pass from point to point along the length of the rod, so that we are always applying the standard theory to a portion of the body near the origin. Any truly nascent, that is to say infinitesimal, strain would be consistent with nascent angular displacement at all points, but this does not deal with all the practical problems in which the strains are merely very small fractions.

213. There are several quadric surfaces which may be used to give graphical representations of a strain. The only one which has been used in the previous pages is the inverse strain ellipsoid. This must be distinguished from a less useful surface which is called the strain ellipsoid, namely the ellipsoid in the strained figure which corresponds to a sphere of unit radius in the unstrained figure.

Some of the surfaces which are occasionally used belong only to the theory of small strain. One of these, which is worth notice, may be called the elongation

quadric. This is the pair of quadric surfaces whose equation is

$$e_1x^2 + e_2y^2 + e_3z^2 + \phi_1yz + \phi_2zx + \phi_3xy = \pm k,$$

where k is a small number of the same order of magnitude as the components of strain. It is a surface, in the unstrained figure, which is either a single ellipsoid or else a pair of conjugate hyperboloids, and has a real central radius in every direction, and whose principal axes are the principal axes of the strain. Let ρ be the length of its central radius in the direction (l, m, n), then the elongation, $\frac{R-r}{r}$, of a line in this direction is k/ρ^2, or $-k/\rho^2$, according as the upper or lower sign is adopted in the equation of the surface. Here the notation of § 200 is used; R being the strained length of a line of length r in the direction (l, m, n). This result is immediately derivable from the equation of the inverse strain ellipsoid for small strain, namely

$$(1 + 2e_1)\,x^2 + (1 + 2e_2)\,y^2 + (1 + 2e_3)\,z^2 + 2\phi_1yz + 2\phi_2zx + 2\phi_3xy = 1;$$

for this gives

$$e_1l^2 + e_2m^2 + e_3n^2 + \phi_1mn + \phi_2nl + \phi_3lm = \tfrac{1}{2}\left(\frac{R^2}{r^2} - 1\right) = \frac{R-r}{r}\cdot\frac{R+r}{2r},$$

and in the limiting case of small strain $\frac{R+r}{2r}$ becomes equal to 1. It will be noticed that, for any strain, the directions of zero elongation, at a given point, if they exist, are on a quadric cone.

All the surfaces that are used in this way depend only on the strain, and are independent of the choice of coordinate axes. Thus we get relations between the components of strain, referred to coordinate axes in various directions, from the invariants for a change of the directions of the axes which are given in § 103.

CHAPTER XVI

STRESS

214. The mutual force action between adjacent parts of a body is called stress. Pressure in a fluid is the simplest case of stress. Hydrostatics is therefore a convenient starting point for the consideration of the measurement of stress of a more general character.

A force represented by pressure in a fluid is a surface force; that is to say it has a finite value per unit area of a surface across which it acts. It differs essentially from a body force, like weight, which has, at each point of a body to which it is applied, a finite value per unit volume, or unit mass, of the body. In the case of equilibrium of an elementary portion of fluid, whose linear dimensions are represented by ϵ, body forces proportional to ϵ^3 are balanced by the surface forces, due to pressure on the surface of the element, which are proportional to ϵ^2. And in the limiting case, of ϵ tending to zero, a body force tends to zero in comparison with the surface forces, but has a finite ratio to the differences between the opposing surface forces exerted on opposite sides of the element.

Accordingly we find, in hydrostatics, by the consideration of forces of the order ϵ^2, that pressure at any point has a single value, the same across surfaces in all directions at that point. And by the consideration of forces of the order ϵ^3, we get the fundamental equations of the subject, namely

$$\frac{\partial p}{\partial x} = \rho X, \quad \frac{\partial p}{\partial y} = \rho Y, \quad \frac{\partial p}{\partial z} = \rho Z,$$

where p is the pressure, ρ the density of the fluid, and X, Y, Z are the components, in the directions of the axes, of the body forces acting on the fluid per unit mass.

These equations give, for example, the result that the pressure at any point in a heavy liquid, with a horizontal surface at which pressure is zero, would be proportional to the depth.

215. *Solid Body*. In the case of a solid body a similar procedure can be followed. We have body forces as before, and surface forces representing stress. But the stress at a given point is not, in general, represented by a single pressure.

Take any point, P, within the body, and draw at this point a small portion of surface, of area α, and a straight line PN, normal to the surface, with direction cosines (l, m, n); and consider the surface forces acting over this area on the portion of the body which is bounded by this surface, and is on the side of it away from N.

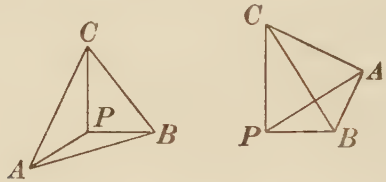
In hydrostatics, in the limiting case in which α is infinitesimal, the force would be $-p\alpha$ in the direction (l, m, n), where p is the pressure at the point P. In the case of a solid body the corresponding force has some positive magnitude, $F\alpha$, and direction cosines (λ, μ, ν). Accordingly a quantity with magnitude F, and direction (λ, μ, ν), is called the stress across the area in question; and if this is known for all directions (l, m, n), the state of stress in the body at the point P is known. The way in which the values of F and (λ, μ, ν) for different directions, (l, m, n), are related to one another, will now be investigated.

216. *Components of Stress*. At the given point P draw an elementary tetrahedron, three of whose faces, PBC, PCA and PAB, are parallel to the coordinate planes, while the fourth face, ABC, represents the elementary area, α, and (l, m, n) are the direction cosines of the normal to this face outwards; and first take l, m, n all positive, so that the areas of the faces are α, $l\alpha$, $m\alpha$, $n\alpha$ respectively. Let us find the conditions of equilibrium of this

elementary tetrahedron, under the action of the surface forces on its faces; in comparison with which any body force, with a finite value per unit volume, is ultimately zero. Let

$$X_1,\ Y_1,\ Z_1,\quad X_2,\ Y_2,\ Z_2,\quad X_3,\ Y_3,\ Z_3$$

be the components, in the directions of the axes, of the stresses across the three faces of the tetrahedron which



are parallel respectively to the planes of yz, zx and xy. Then resolution in the directions of the axes gives

$$F\lambda\alpha - X_1 l\alpha - X_2 m\alpha - X_3 n\alpha = 0,$$
$$F\mu\alpha - Y_1 l\alpha - Y_2 m\alpha - Y_3 n\alpha = 0,$$
$$F\nu\alpha - Z_1 l\alpha - Z_2 m\alpha - Z_3 n\alpha = 0.$$

The signs in these equations indicate the directions in which the forces on the tetrahedron, due to stress, are reckoned to act. Dividing by α we get a result which is correct in the limiting case in which α tends to zero, the tetrahedron shrinking to a point, namely

$$\lambda F = lX_1 + mX_2 + nX_3,$$
$$\mu F = lY_1 + mY_2 + nY_3,$$
$$\nu F = lZ_1 + mZ_2 + nZ_3.$$

It can easily be seen that we have the same equations when l, m and n are not all positive. Suppose for example that l is negative, as shown in the alternative figure in the diagram; the area of one of the faces of the tetrahedron is then $-l\alpha$, but the components of stress across this face would be taken to be $-X_1$, $-Y_1$, $-Z_1$.

Three relations can now be found, connecting the nine

components, X_1, X_2 ......, by considering the conditions of equilibrium of an elementary cube, in the body, with its edges parallel to the axes. Let ϵ be the length of an edge of the cube. The stress across each of the six faces of the cube is represented by three components, one of them normal to the face, and the other two tangential. And in the limit, when ϵ tends to zero, the corresponding tangential forces acting on two opposite faces of the cube form two couples, in planes parallel to two of the co-ordinate planes. Thus we have altogether six couples acting on the cube. And as they cannot be balanced by body forces, they must balance one another. The two couples which are in planes parallel to the plane of yz have moments $Y_3\epsilon^2 \times \epsilon$ and $-Z_2\epsilon^2 \times \epsilon$, therefore $Y_3 = Z_2$. Similarly $Z_1 = X_3$ and $X_2 = Y_1$. Accordingly the number of independent components of stress is reduced to six.

For the six independent components of stress let us adopt the notation p_1, p_2, p_3, t_1, t_2, t_3, so that the equations for F and (λ, μ, ν) are written

$$\lambda F = lp_1 + mt_3 + nt_2,$$
$$\mu F = lt_3 + mp_2 + nt_1,$$
$$\nu F = lt_2 + mt_1 + np_3,$$

p_1, p_2, p_3 being normal components, and t_1, t_2, t_3 tangential components. These three equations give the statement, in the theory of stress in a solid body, which corresponds to the statement in hydrostatics that the stress consists of a single pressure, the same across all surfaces at a given point. It should be noticed that the normal components, p_1, p_2, p_3, are defined so as to represent tensions when they are positive.

217. *Graphical representation of Stress.* The equations

$$\lambda F = lp_1 + mt_3 + nt_2,$$
$$\mu F = lt_3 + mp_2 + nt_1,$$
$$\nu F = lt_2 + mt_1 + np_3$$

show that the state of stress, at any given point in the body, has a certain symmetry which can be exhibited graphically by means of the pair of quadric surfaces represented by the equation

$$(p_1x^2 + p_2y^2 + p_3z^2 + 2t_1yz + 2t_2zx + 2t_3xy)^2 = 1,$$

which may be written

$$p_1x^2 + p_2y^2 + p_3z^2 + 2t_1yz + 2t_2zx + 2t_3xy = \pm 1.$$

This pair of surfaces is either a single ellipsoid, or else a pair of conjugate hyperboloids. It is convenient to regard it, in either case, as a single surface, and to call it, for the sake of brevity, the stress quadric. It has a central radius in all directions, except those given by the asymptotic cone when this exists.

Let r be the length of the central radius, OQ, drawn in the direction (l, m, n), and p the length of the perpendicular from the centre on the tangent plane at Q. The equation of this tangent plane is

$$r(lp_1 + mt_3 + nt_2)x + r(lt_3 + mp_2 + nt_1)y$$
$$+ r(lt_2 + mt_1 + np_3)z + d = 0,$$

or $$rF(\lambda x + \mu y + \nu z) + d = 0,$$

where d has one of the two values $+1$ and -1. Therefore $F = \dfrac{1}{rp}$, and the direction (λ, μ, ν) is the direction of the perpendicular from the centre on the tangent plane if $d = -1$, and the opposite direction if $d = 1$.

Let us assume that the stress does not possess the special symmetry which would make the stress quadric a surface of revolution. The surface has then only one set of principal axes, which are the lines along which the directions (l, m, n) and (λ, μ, ν) coincide, so that the stress is wholly normal. This is a unique set of lines; and as they have a specification in terms of stress, they are independent of the choice of directions of the coordinate axes. They are called the principal axes of the

stress. If they are chosen as coordinate axes, the equation of the stress quadric is

$$Ax^2 + By^2 + Cz^2 = \pm 1,$$

where A, B and C are the normal components of stress across the principal planes, and the tangential components are zero. This shows that the stress quadric at any given point is the same surface whatever choice is made of directions for the coordinate axes.

Accordingly the theory of quadric surfaces provides the result that, at each point of a body subject to stress, three planes at right angles can be found across each of which the stress is normal. The stresses across these planes are called the principal normal stresses at the point in question.

If the stress is symmetrical about a straight line this conclusion is not affected. The only result is that we have a certain variety of directions for principal axes of stress.

The existence of principal axes of stress having been established, other quadric surfaces may be used to represent graphically, in some different way, the arrangement of stress at any given point of a body. Take, for example, the surface whose equation referred to principal axes of stress is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1.$$

This may be called the stress ellipsoid. If p is the length, and (λ, μ, ν) are the direction cosines, of the perpendicular from the centre on a tangent plane of this ellipsoid,

$$p^2 = A^2\lambda^2 + B^2\mu^2 + C^2\nu^2.$$

Therefore p represents the magnitude of the stress across a plane parallel to this tangent plane at the point in question.

It should be noticed here, as elsewhere, that in the case of figures introduced only for graphical representa-

tion, the scale on which they are drawn is a matter of no consequence; so it is not necessary to pay attention to units.

All the formulae relating to central quadric surfaces are at our disposal, for interpretation in terms of stress, so far as we have a use for them. For example, we have the invariants of § 103. These, applied to the stress quadric, give the important results

$$p_1 + p_2 + p_3 = A + B + C,$$

$$p_2 p_3 + p_3 p_1 + p_1 p_2 - t_1^2 - t_2^2 - t_3^2 = BC + CA + AB,$$

$$p_1 p_2 p_3 + 2 t_1 t_2 t_3 - p_1 t_1^2 - p_2 t_2^2 - p_3 t_3^2 = ABC.$$

The point to be observed is that the existence of this surface, independently of any particular choice of coordinate axes, has been established.

218. *Introduction of Body Forces.* To find the effect of body forces, let us first take the simple case of hydrostatics. The natural procedure is to find the conditions of equilibrium of a portion of the fluid in the form of a cube, with edges parallel to the axes, in the limiting case in which the length, ϵ, of an edge of the cube tends to zero. Let ρ be the density of the fluid, p the pressure, and X, Y, Z the components, in the directions of the axes, of the body force per unit mass at the point in question. As we are concerned only with the limiting case, the components of body force acting on the cube may be written $X\rho\epsilon^3$, $Y\rho\epsilon^3$, $Z\rho\epsilon^3$; and similarly the forces in the direction of the axis of x due to stress are $p\epsilon^2$ on one face of the cube, and $-\left(p + \frac{\partial p}{\partial x}\epsilon\right)\epsilon^2$ on the opposite face. Therefore for equilibrium $\frac{\partial p}{\partial x} - \rho X = 0$; and similarly $\frac{\partial p}{\partial y} - \rho Y = 0$, $\frac{\partial p}{\partial z} - \rho Z = 0$.

In the case of a solid body the same procedure gives equations of similar form, but containing more terms,

because all the six components of stress have to be taken into account. The forces due to stress which act on the cube, in the direction of the axis of x, are now

$$-p_1\epsilon^2,\ \left(p_1+\frac{\partial p_1}{\partial x}\epsilon\right)\epsilon^2 \text{ on faces parallel to plane of } yz;$$

$$-t_3\epsilon^2,\ \left(t_3+\frac{\partial t_3}{\partial y}\epsilon\right)\epsilon^2 \quad ,, \quad ,, \quad ,, \quad zx;$$

$$-t_2\epsilon^2,\ \left(t_2+\frac{\partial t_2}{\partial z}\epsilon\right)\epsilon^2 \quad ,, \quad ,, \quad ,, \quad xy.$$

Therefore resolution in the direction of the axis of x gives

$$\frac{\partial p_1}{\partial x}+\frac{\partial t_3}{\partial y}+\frac{\partial t_2}{\partial z}+\rho X=0;$$

and similarly, resolutions in the directions of the other axes give

$$\frac{\partial t_3}{\partial x}+\frac{\partial p_2}{\partial y}+\frac{\partial t_1}{\partial z}+\rho Y=0,$$

$$\frac{\partial t_2}{\partial x}+\frac{\partial t_1}{\partial y}+\frac{\partial p_3}{\partial z}+\rho Z=0.$$

There are no more equations, because the equations obtained by taking moments have already been used to reduce the number of components of stress from nine to six.

These three equations are the fundamental equations which give the conditions of equilibrium of a solid body in the theory of elasticity; the six components of stress having been expressed in terms of the six components of small strain, namely e_1, e_2, e_3, ϕ_1, ϕ_2, ϕ_3, (§ 210), by means of the experimental relations between stress and strain which are sometimes referred to as Hooke's law.

CHAPTER XVII

VECTOR DISTRIBUTIONS

219. Certain physical quantities can be represented graphically by systems of lines or surfaces. A line means, in general, a curved line. A line which merely defines a route from one point to another may be called a path.

A quantity which has a certain value at each point of a given region may be said to have a certain distribution in that region; the term distribution denoting the aggregate of its values, each located at the point to which it belongs.

A given region of space will usually be assumed to be such that any point of it can be reached from any other point of it by paths within the region. Two different paths joining two given points are said to be reconcilable if one can be shifted gradually into coincidence with the other, without passing out of the region. A region in which any two paths, between any two given points, are reconcilable, is said to be simply connected. Thus the whole of space, and the region inside a sphere, and the region outside a sphere, are examples of simply connected regions. But such regions as that inside a ring, or outside an infinite circular cylinder, do not possess this property, and are examples of multiply connected regions.

220. *Scalar and Vector quantities*. A quantity which is specified by a magnitude, that is to say by a single number, is called a scalar quantity. If it has a certain value, (or set of alternative values), at each point, it may be called a point function, being a function of whatever coordinates of a point are used. Its distribution may be called a scalar distribution. For example, the distribution of density in a body is a scalar distribution.

A quantity which is specified by a magnitude and a direction, and obeys the parallelogram law of composition and resolution*, is called a vector; and its distribution may be called a vector distribution. For example, the distribution of velocity in a moving fluid is a vector distribution.

In the graphical representation of a vector, the direction is taken so that the magnitude is positive. But in the corresponding algebraical formulae, the distinction between this specification and the equivalent one, by a negative magnitude and the opposite direction, becomes unnecessary, and is dropped. This is a familiar feature in the specification of a velocity or a force in mechanics.

The symbols of ordinary algebra will be used here, so that if a vector is referred to as the vector F, F is to be understood to be its magnitude, in accordance with the ordinary usage of mechanics.

A vector can be specified by the magnitudes of its components in the directions of given coordinate axes, that is to say by three scalar quantities.

The distribution of a vector is sometimes called its field, as in the expressions a field of force, and a field of induction.

A scalar quantity whose distribution is studied will be assumed, in general, to be a continuous function of coordinates, and to possess differential coefficients with regard to them, and to have only a single value at each point. It may exceptionally have more than one value at each point, or be discontinuous over a surface or at a single point. But discontinuity throughout any volume is excluded. A vector whose distribution is studied will be assumed to be such that the magnitudes of its components, in the directions of the axes, satisfy these conditions.

* In various particular cases the parallelogram law of composition is practically obvious, or a matter of definition. A proof of it sufficiently general to indicate, with some degree of clearness, what it depends on, was given by Laplace in the *Mécanique Céleste*. A good and simple form of such a proof, by Mr W. E. Johnson, will be found in *Nature*, vol. 41, p. 153.

Any given portion of surface will always be assumed to have two distinct sides*, and usually to have a normal at each point. The two sides of a surface may be distinguished arbitrarily as the side A and the side B. But the terms inside and outside may also be used to distinguish the two sides; and in the case of a surface which is defined as closed these terms have their natural meanings.

221. *Scalar Distributions.* The distribution of a scalar quantity, ϕ, can be represented graphically by a system of surfaces, over each of which ϕ is constant, drawn for equal small intervals of the value of ϕ. They are called level surfaces, or equipotential surfaces if ϕ is a potential. If the surfaces are conceived to be sufficiently numerous, the number cut by any line, per unit length of it, at any point, represents the rate at which ϕ increases or decreases along that line at that point. At any given point, let ϵ be the distance between consecutive surfaces measured along the normal; then the distance between them, along a line inclined to the normal at an angle θ, becomes $\epsilon \sec \theta$ when the surfaces are sufficiently numerous. Thus if F is the rate of increase of ϕ along the normal, $F \cos \theta$ is its rate of increase along the inclined line.

Contour lines on a map, conceived to be sufficiently numerous, furnish an example, in one plane, of this method of representation of a scalar quantity. In this case the quantity represented is the height of the ground above some datum.

222. *Vector Distributions.* A vector distribution can be represented graphically by a system of lines. Take any point A in the given region, and draw a straight line AB in the direction of the vector at A; and from B draw

* A model of a surface which has not got two distinct sides may be made by giving a strip of paper a single twist, and then pasting the two ends together.

BC in the direction of the vector at B; and from C draw CD in the direction of the vector at C. Proceeding in this way, we get a polygon ABC......; and in the limiting case in which the lengths of the sides of the polygon tend to zero, we get a curved line whose tangents, in the direction in which the line is drawn, give the direction of the vector at each point of it. Any number of such lines can be drawn, starting from any points, and of such lengths as may be desired. They cannot intersect, except where the vector vanishes; and they may be conceived to be so numerous that they represent the distribution of direction completely. Their existence is verified by the theory of the differential equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

where X, Y and Z are given functions of x, y and z, which may be interpreted as the components of a vector.

Let us now make these lines represent also the magnitude of the vector. We have the liberty of choosing where each line begins, and how long it is; and we can eliminate some lines, or add others, without disturbing the representation of direction. Thus the lines can be arranged so that the number per unit area cutting a small piece of surface, drawn at right angles to them, at any point, is proportional to the magnitude at that point. The lines so arranged represent the vector distribution completely. They form a fibrous structure, with an indication on each fibre of its direction.

A further step is now taken to make the representation more definite. That is to say we agree to count the lines not singly but by packets, in the way any number may be expressed by dozens and a fraction of a dozen. And the number of single lines that go to a packet is defined as that which represents a unit of magnitude. Reckoned in this way, any given number of lines is equal to the quantity which it represents, instead of being only proportional to it, and is not in general a whole number.

The word packet does not enter into any statement, we merely agree to count lines in this way. Surfaces representing a scalar quantity may be counted in the same way.

It may happen that all the lines are continuously drawn, so that they are either closed curves or extend to the boundary of the region, or can be given this character by joining up beginnings and endings occurring at the same point. But in general we shall have, scattered through the fibrous structure, a great number of loose beginnings and endings. The excess, whether positive or negative, of the number of lines which begin over the number which end, in any given volume, is called the total divergence of the vector in that volume. And this excess per unit volume, at any given point, is called the divergence at that point. Thus divergence has the same relation to total divergence that density has to mass. It has a certain scalar distribution.

A vector distribution is said to be solenoidal in any region throughout which its divergence is zero, so that it can be represented by unbroken lines. The word solenoidal signifies tubular; and the word tubular may be used if it is preferred.

Electromagnetic induction owes its introduction into physics, and its great importance, partly to the fact that its distribution is everywhere solenoidal. If lines of induction are found anywhere, they are either closed curves or must go somewhere. And one of the objects, in the design of such an apparatus as a dynamo, is to keep them where they are useful, instead of letting them stray away.

223. *Flux.* Let the two sides of a surface be distinguished as the side A and the side B. The number of lines which, taking account of the direction in which they are drawn, cut a given portion of surface, say from A to B, will be understood to mean the net number; that is to say the excess of the number which cut it from A

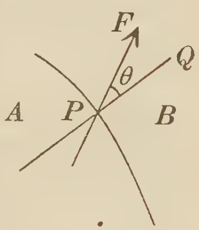
to B over the number which cut it from B to A. This is called the total flux of the vector across the surface from A to B. It may be either positive or negative. Also the net number per unit area at any point of the surface is called the flux of the vector at that point, across the surface from A to B.

It is obvious from this that if the flux across a surface from A to B is f, the flux across it from B to A is $-f$. Also that the total flux across a closed surface outwards is equal to the total divergence in the enclosed volume. This may be either positive or negative. By the use of the term flux the awkwardness of introducing the idea of a negative number of lines is avoided.

For the calculation of flux, it is useful to draw a thin tube, enclosing certain lines without cutting them; the meaning of the term "thin" being that we take the limiting case when the tube is shrinking to a thread.

If the vector distribution is solenoidal, the total flux is the same across all sections of the tube. And in all cases, the total flux is the same across all sections at any given point of the tube, because all these sections cut the same lines.

To find the flux across a given surface, draw a thin tube, in this way, through a point P on the surface. Let F be the positive magnitude of the vector at P, and let the sides, A and B, of the surface, be chosen so that the direction of the vector is from the side A to the side B. Then the normal, PQ, of the surface, drawn from A to B, makes an acute angle with the direction of F. Let this angle be θ, and let G be the resolved part of F in the direction PQ, then $G = F\cos\theta$.



Let α be the area of the cross section of the tube at P, then the tube contains αF lines at P. Now the area of the section of the tube by the surface is $\alpha\sec\theta$. Therefore the flux across the surface, from A to B, at the

point P, is the ratio of αF to $\alpha \sec\theta$, that is to say $F\cos\theta$, which is equal to G. Accordingly the flux across the surface, from B to A, at the point P, is $-G$.

This shows that a single statement is sufficient, namely that if the sides, A and B, of a surface are chosen arbitrarily, and the resolved part of the vector in the direction of the normal from A to B at a given point, P, is denoted by N, the flux across the surface at this point from A to B is equal to N.

Let (l, m, n) be the direction cosines of the normal at P from A to B, and let u, v, w be the components of the vector at this point, then

$$N = lu + mv + nw;$$

and the total flux across the surface is

$$\iint (lu + mv + nw)\, dS,$$

where dS is an element of area of the surface. Therefore, for a given vector distribution, and a given surface, the value of this expression is independent of the choice of coordinate axes.

224. *Composition of Vector Distributions.* Two or more vector distributions, in the same region, can be compounded by compounding the vectors at each point; and the distribution of the resultant vector may be called the resultant of the given distributions. Now the sum of the resolved parts of two or more vectors, in any direction, is equal to the resolved part of their resultant in that direction. Therefore the flux of the resultant distribution, across any surface, is equal to the sum of the fluxes of the component distributions across the same surface. This shows that, in order to find the flux of the resultant distribution, it is not necessary to construct the lines of this distribution and count them; for it can be found by merely counting all the lines of all the component distributions. Also the divergence of the resultant distribution, at any point, is equal to the alge-

braical sum of the divergences, at that point, of the component distributions; because the total divergence in any volume is equal to a certain flux, namely the total flux across the bounding surface. This shows, for example, that the resultant of two or more solenoidal distributions is itself solenoidal.

The thing which is not obtained immediately, from this superposition of systems of lines, is a representation of the direction of the resultant vector. But this can be calculated from the fluxes across three planes at right angles to one another.

Each of the components, u, v, w, of a given vector, F, in the directions of the axes, is a scalar quantity, a function of the coordinates. But each component may also be regarded as a vector with direction parallel to one of the axes. And the resultant of the vector distributions of u, v and w is the distribution of F; and the sum of their divergences at any point is the divergence of F at that point. Consider the vector $(u, 0, 0)$ represented by lines parallel to the axis of x, drawn in the direction of this axis where u is positive, and in the opposite direction where u is negative. For this distribution, a thin tube, drawn as before to enclose certain lines, is a cylinder parallel to the axis of x. The area, α, of its cross section, is the same at all points along it. The total divergence between two cross sections, at points (x, y, z) and $(x+\delta x, y, z)$, is $\left(u+\frac{\delta u}{\delta x}\delta x\right)\alpha - u\alpha$, and the volume enclosed between these sections is $\alpha\,\delta x$; therefore the divergence at any point is $\frac{\partial u}{\partial x}$. Similarly the divergences of the vectors $(0, v, 0)$ and $(0, 0, w)$ are $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$. Therefore the divergence of F is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}.$$

The value of this expression is therefore independent of the choice of axes.

This shows that the condition that the distribution of F is solenoidal in a given region is that the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied throughout that region.

225. *Line Integral and Circulation.* Draw any path from a point A to a point B; and let R be the resolved part of the vector along the tangent in the direction of the path, at any point of it. Then $\int Rds$ from A to B is called the line integral of the vector along this path, ds being an element of length of the path. It is equal to

$$\int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right)ds,$$

which may be written

$$\int(u\,dx + v\,dy + w\,dz).$$

For a given vector distribution, and a given path, the value of this is independent of the choice of axes. If the path is a complete circuit, A and B coinciding, the line integral is called the circulation of the vector in this circuit, in the direction chosen. Circulation in the opposite direction has the same value with its sign changed.

226. *Lamellar Distribution.* A given vector, F, specified by its components, u, v, w, is said to have a lamellar distribution, in a given region, if it varies from point to point so that

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

This means that u, v, w can be written in the form

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z},$$

where ϕ is a scalar quantity whose variations from point to point are thus specified. This relation between F and ϕ is expressed by saying that F is the gradient (or slope)

of ϕ. And a vector with a lamellar distribution is called a gradient vector. Also ϕ is called the potential of F; thus a gradient vector is a vector which has a potential.

The increase of ϕ per unit length along any path, at any point of it, is equal to the resolved part of F at the same point in the same direction, for each is equal to

$$u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds},$$

ds being an element of the path. Accordingly the relation of ϕ to F, and consequently the condition for a lamellar distribution in terms of u, v and w, is independent of the choice of coordinate axes.

Let the surfaces over each of which ϕ is constant be drawn for equal small intervals, $\delta\phi$, of the value of ϕ. The direction cosines of a normal to a surface are proportional to u, v, w (§ 167), thus the surfaces cut the lines of the vector orthogonally. They cannot intersect except possibly at points at which F is zero. And they cannot have loose edges, so they are closed surfaces or else extend to infinity or to the boundary of the region. Thus they form a laminated structure in the given region except in any volume in which F is zero. The magnitude of the vector, at any point, is represented by the number of surfaces per unit length cut by a line at right angles to them at this point; and is equal to this number if the surfaces are counted by packets, the number of single surfaces to a packet being defined as that which represents a unit of magnitude. Similarly the number per unit length cut by any line, at any point of it, is equal to the positive magnitude of the resolved part of the vector along that line. A given lamellar vector distribution specifies these surfaces, and something more, namely the marks to be attached to them to indicate which way across them corresponds to increase of ϕ. The system of surfaces then represents the vector distribution.

It should be noticed that the measurement of the resolved part of the vector in any given direction by counting surfaces agrees with the parallelogram law; for the ratio of the number of surfaces cut, per unit length, in an inclined direction to the number cut, per unit length, in the normal direction, is the cosine of the angle between the two directions.

227. *Potential.* Let us now seek the values of ϕ. We only know the way it varies, so we must start with an arbitrary value for it at some one point; and if this leads to a single value for it at each point of the region, ϕ is said to be single valued. The definition of ϕ shows that the line integral of F along any path is equal to the total increase of ϕ along this path; so ϕ is single valued if, and not unless, the line integral of F from any given point A to another point B has the same value for all paths from A to B. Take any path from A to B, and shift it very slightly so that we get a second path between these points cutting the same surfaces. Thus we have the same series of values of ϕ along the second path as along the first, though their distribution is presumably slightly varied. By the same procedure, applied to the second path, we can get a third with the same property, and from this a fourth, and so on. Therefore the line integrals of F along any two paths from A to B are the same if each path can be shifted gradually into coincidence with the other without going outside the given region, that is to say if the paths are reconcilable (§ 219). Therefore in any simply connected region ϕ is single valued, and the circulation of F in any circuit is zero. A value for ϕ at some one point, O, having been chosen, its values at all other points are found uniquely by means of the line integrals along paths from O.

But in a multiply connected region, any two points can be joined by irreconcilable paths, the line integrals along which may be unequal; and circuits can be drawn

the circulations in which may not be zero. Therefore, in a multiply connected region, we may have a lamellar vector distribution for which, although F is single valued, ϕ must be permitted to have a number of alternative values at each point.

It will now be shown that, in these cases, the values of ϕ can always be classified in terms of the circulations of F in the various circuits, irreconcilable with one another, which the region provides.

Take, for example, the case in which the given region is that which is outside a circular cylinder. The circulations in all circuits which do not embrace the cylinder are zero. But the circulations in circuits which embrace the cylinder once may have a value other than zero, the same for them all, say α or $-\alpha$ according to the direction in which the circuit is traversed. And then the circulation in a circuit which goes round the cylinder n times is $n\alpha$ or $-n\alpha$. Therefore taking account of all the paths, irreconcilable with one another, which can be drawn from a point O to a point A, it is clear that, if ϕ_A is a value of ϕ at A, the general formula for the values which ϕ may assume at A is $\phi_A \pm n\alpha$, where n is a positive integer or zero. Similarly if the region provides several independent irreconcilable circuits, we get a corresponding formula. If there are three such circuits with circulations α, β, γ, the formula is

$$\phi_A \pm n\alpha \pm m\beta \pm r\gamma,$$

where n, m and r are positive integers or zero.

In a multiply connected region the series of surfaces, drawn for equal intervals $\delta\phi$ of the value of ϕ, may be confused by the circulations α, β...... in irreconcilable circuits not being exact multiples of any interval. But this does not matter because, when $\delta\phi$ is infinitesimal, a single series can be obtained by permitting one, or several, of the intervals to be less than $\delta\phi$, without the character of the series being affected.

If two vectors possessing potentials, ϕ, ϕ', are compounded, their resultant is a vector with potential $\phi + \phi'$. Therefore the resultant of any number of lamellar vector distributions is itself lamellar.

For some purposes the potential of a gradient vector is defined by the equations

$$\frac{\partial \phi}{\partial x} = -u, \quad \frac{\partial \phi}{\partial y} = -v, \quad \frac{\partial \phi}{\partial z} = -w,$$

the vector being its rate of decrease. For example, magnetic potential is derived from magnetic force in this way. Which definition is adopted in any particular case is a matter of no consequence, involving only the sign of ϕ.

228. *Curl.* If F is any vector whose components are u, v, w, the vector F', whose components are

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

is called the curl of F, and F is called the vector potential of F'. Thus a gradient vector may be defined as a vector whose curl is zero at all points. Also the distribution of curl is solenoidal, for its components satisfy the test for this.

Now the differential equation of the surfaces which cut the lines of the vector F orthogonally, if they exist, is

$$u\,dx + v\,dy + w\,dz = 0,$$

because the meaning of this equation is that tangents of the surface at any point are at right angles to F at that point. And we know, from the theory of this differential equation, that it represents a system of surfaces if, and not unless,

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

that is to say if, and not unless, the curl of F is either zero, or at right angles to F. So this is the condition

that a system of surfaces can be drawn cutting the lines of F orthogonally. If F' is not zero, but is at right angles to F, portions of the surfaces can be arranged so as to form a system representing the distribution of F; but they would have loose edges. They would no longer be restricted to being closed surfaces or extending to the boundary of the region.

A simple example of this is afforded by the vector whose components are u, 0, 0. This is not a gradient vector unless $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial y}$ are both zero, but it is at right angles to its curl. And it can be represented by surfaces with loose edges, which are portions of planes at right angles to the axis of x.

229. *Stokes's Theorem.* Thus we see that the curl of a given vector, F, though defined with reference to coordinate axes, has properties which are independent of the choice of axes. That it is altogether independent of the choice of coordinate axes, except as to its sign, which is a matter of arbitrary choice, is proved by Stokes's theorem. This is a well known proposition of integral calculus which is expressed by the equation

$$\iint\{l\,(w_2 - v_3) + m\,(u_3 - v_1) + n\,(v_1 - u_2)\}\,dS$$
$$= \int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right) ds.$$

It is assumed here that we are dealing with a portion of surface for which dS is an element of area, bounded by a circuit of which ds is an element of arc. The direction cosines of the normal of the surface are (l, m, n). And u, v, w are any set of three single valued functions of the coordinates, whose nine partial differential coefficients, with regard to x, y and z respectively, namely $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \ldots\ldots$, are denoted by u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3. The double integral extends over the surface, and the single

integral extends round the circuit in the direction which makes a right-handed or a left-handed screw with the direction of the normals, according to the way the coordinate axes are arranged*.

Thus if we interpret u, v, w as the components of the vector F, we obtain the result that the total flux of the curl of F across any portion of surface is equal to the circulation of F in the boundary of the surface, and is therefore independent of the choice of coordinate axes.

This proposition can be applied to a surface which is bounded by two or more circuits, by making cuts arranged so as to reduce it to the form of a surface bounded by only one circuit. A circulation of the vector in this one circuit then consists of circulations in the given circuits in such directions that the portions of the

* The general proof of Stokes's theorem being rather complicated, it is worth while to note that the proof for a surface such that l, m and n are positive, and the bounding circuit is cut by planes parallel to the coordinate planes in not more than two points, is very simple. The terms involving u, v and w can be taken separately. To prove the equation

$$\iint (mu_3 - nu_2)\, dS = \int u \frac{dx}{ds}\, ds$$

for this case, take x and y as independent variables specifying a point on the surface, z and u being treated as functions of x and y. Then $\frac{\partial z}{\partial y} = -\frac{m}{n}$, and

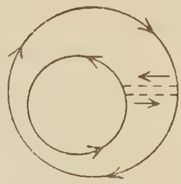
$$\frac{\partial u}{\partial y} = u_2 + u_3 \frac{\partial z}{\partial y} = u_2 - \frac{m}{n} u_3;$$

and $\frac{1}{n} dx\, dy$ represents an element of area of the surface. Therefore

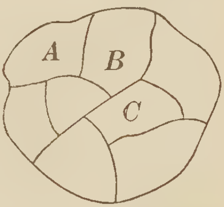
$$\iint (mu_3 - nu_2)\, dS = -\iint \frac{\partial u}{\partial y}\, dx\, dy = -\int (u'' - u')\, dx,$$

where u' and u'' are the values of u at the extremities of a strip of the surface between consecutive planes parallel to the plane of yz. Assuming the axes to be drawn in the standard way, so that the proposed direction of circulation makes a right-handed screw with the axis of z, u' belongs to the point in the circuit at which $\frac{dx}{ds}$ is positive, and u'' belongs to the point at which $\frac{dx}{ds}$ is negative. Therefore $-\int (u'' - u')\, dx$ is equal to $\int u \frac{dx}{ds} ds$ for the whole circuit. The corresponding results involving v and w can be obtained in the same way, y and z, or z and x being taken as independent variables. Thus the required formula is proved.

circulation in each cut cancel one another. The diagram shows a portion of surface bounded by two circles; and a cut which converts the boundary into a single circuit is shown by dotted lines. And the sum of the circulations of the vector in opposite directions in the two circles is equal to the flux of its curl across the surface.



230. *Network.* On a given portion of surface which has two distinct sides, called inside and outside, and is bounded by a circuit, draw a network of paths. And let us call the portions into which the surface is divided "meshes," and the points at which paths meet "knots." Each portion of the bounding circuit, between successive knots, belongs to only one mesh, and every other portion of path between two knots belongs to two meshes. Call the meshes $A, B, C \ldots$, and in each mesh draw a normal to the surface outwards. Let $\alpha, \beta, \gamma \ldots$ be the circulations of a given vector, in the circuits of the meshes $A, B, C \ldots$, in the direction which makes a right-handed screw with the direction of the normal outwards. Then



$$\alpha + \beta + \gamma + \ldots\ldots$$

is equal to the circulation in the bounding circuit, because the line integrals for the internal portions of paths cancel one another. Also a surface integral for the whole surface is equal to the sum of the corresponding integrals for the meshes. Thus we have a way of dividing up the circulation and the surface integral into parts which can be added together. Inwards may of course be substituted for outwards, or left-handed for right-handed.

231. *Gravitation.* If the vector whose distribution is studied is the force of gravitation that would be exerted at any point, on a particle of unit mass placed there, due

to a given attracting particle of mass m at a given point A, the lines are straight lines, radiating from A, uniformly distributed in all directions.

Let units be chosen so that the force at a distance r from the point A is m/r^2. Then the total number of lines, counted by packets, is $4\pi m$, because the area of a sphere of unit radius is 4π. There is no divergence except $-4\pi m$ concentrated at the point A. Elsewhere the distribution is solenoidal. Also the distribution is lamellar, the level surfaces being concentric spheres, and the potential being m/r at a distance r from A. Accordingly the circulation in any circuit is zero.

Thus if we have an attracting system, consisting of masses m_1, m_2 ...... at points A, B ...... , and the vector is the resultant force which it exerts on a particle of unit mass, this vector has a lamellar distribution, and has divergence throughout the attracting system: $4\pi m_1$ at A, $4\pi m_2$ at B, and so on. And the total flux across any closed surface is equal to the total divergence within it. This is called Gauss's theorem.

Also the circulation of the vector in any circuit is zero; that is to say the total work done on an attracted particle as it traverses any circuit is zero.

232. *Magnetism*. The magnetization of a body is specified by a vector quantity, I, called the intensity of magnetization. It has a single value at each point of the body, and its distribution may be specified by lines in the body. If the lines are all closed curves the magnetization produces no external effect. But in general there is divergence; that is to say there are a number of beginnings and endings of lines, some distributed within the body, and others distributed over its bounding surface.

At all these beginnings and endings of lines of I magnetic poles are established, which are the centres of magnetic force, negative poles at beginnings, and positive poles at endings. If a single line represents a positive intensity

δI, a pole of strength $-\delta I$ is established at its beginning, and a pole of strength δI at its ending. Let ρ be the divergence of I at any point within the body. And let σ be its surface divergence at any point on the bounding surface, that is to say the excess of the number of lines which begin there over the number which end there, per unit area of surface. Then an element of volume, δv, contains a pole of strength $-\rho\delta v$; and an element of area of surface $\delta\alpha$ contains a pole of strength $-\sigma\delta\alpha$. These poles constitute what has sometimes been called the free magnetism of the body. Any number of other poles may be introduced which cancel one another, a positive pole and an equal negative pole occurring at the same point.

Now the magnitude of the magnetic force due to a single pole is the product of the strength of the pole and the inverse square of the distance from it. Thus the lines of magnetic force due to a positive pole, of strength m, consist of $4\pi m$ straight lines diverging from the pole, uniformly spaced in all directions; and the lines due to a negative pole, of strength $-m$, consist of $4\pi m$ straight lines similarly converging to the pole. And the magnetic force, H, due to any given system of poles, is the resultant of the magnetic forces due to the single poles; that is to say it is represented by the whole system of lines due to all the poles.

Thus the distribution of the magnetic force, H, due to any given system of magnetized bodies, has been derived from the distribution of I. It is lamellar, because it is the resultant of distributions which are obviously lamellar, namely those due to single poles. Therefore the circulation of H in any closed circuit is zero.

Let us now replace the lines of I by lines of $4\pi I$. These are the same as the lines of I except for being 4π times as numerous. And the number of lines of $4\pi I$ which reach a pole is the same as the number of lines which leave it as lines of H. Therefore the lines of $4\pi I$ and the lines of H, combined, form a system of lines with no diver-

gence, and thus represent a vector with a solenoidal distribution. This vector is called the magnetic induction, and is denoted by B. It may be regarded as introduced into the theory of magnetism on account of its useful solenoidal property. But its great importance lies in the fact that it is the link connecting theories of magnetism and electric currents.

It may be noted, with regard to this, that $4\pi I$ may presumably have any distribution, for we have not imposed any restriction on it. And it is the resultant of a solenoidal distribution, namely that of B, and a lamellar distribution, namely that of $-H$.

233. *Magnetic Shell.* A thin sheet of magnetizable substance, magnetized at each point in the direction of the normal to the sheet at that point, is called a magnetic shell. The thickness of the shell is treated for any purpose of calculation as infinitesimal.

The product, at any point of the shell, of the positive intensity of magnetization and the thickness of the shell is called the strength of the shell at that point. A magnetic shell, when referred to without qualification, is understood to be a shell of uniform strength.

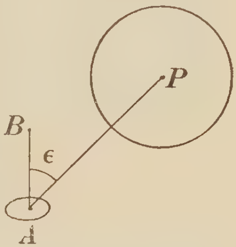
If dS is an element of surface of the sheet, and ϕ is the strength, the shell may be regarded as composed of simple magnets, of moment ϕdS, placed side by side. The two sides of the shell are called the negative side and the positive side; negative poles being distributed over the former, and positive poles over the latter.

Let us assume the well known result, easily proved, that the potential of magnetic force at a point P, due to an elementary magnet, AB, of moment ϕdS, is

$$\frac{\cos\epsilon}{r^2}\phi dS,$$

where r is the distance AP, and ϵ is the angle PAB; A being the negative and B the positive pole.

Applying this to the magnetic shell, of which dS is an element of area, the solid angle (§ 187) subtended at P by this element is obviously $\frac{\cos\epsilon\, dS}{r^2}$ when $\cos\epsilon$ is positive, and $-\frac{\cos\epsilon\, dS}{r^2}$ when $\cos\epsilon$ is negative. Accordingly, if we denote this solid angle by $d\omega$, the potential of magnetic force at P due to this element is $\phi d\omega$ if a radius drawn from P meets the shell on its positive side, and is $-\phi d\omega$ if this radius meets the shell on its negative side.



Therefore, if all radii drawn from P to the shell meet it only on its positive side, the potential at P due to the whole shell is $\phi\Omega$, where Ω is the solid angle subtended at P by the shell. And if all radii meet it only on its negative side the potential is $-\phi\Omega$. And if some parts of the shell have their positive side towards P, and other parts their negative side, the potential is $\phi\Omega$, where Ω may be called the net solid angle, the result of subtracting the solid angles reckoned as negative from those reckoned positive. And generally, it appears that the potential at a given point depends in some fashion on the curve which forms the contour of the shell, and not on the shape of the shell in other respects.

Consider the case of a closed shell. In this case the potential at every point outside it is zero, because, whatever its shape may be, each radius from P cuts it an even number of times, so that the solid angles reckoned as negative cancel those reckoned as positive. And similarly, the potential at every point inside it is $\pm 4\pi\phi$, because all radii from P meet it, and each of them cuts it an odd number of times. The plus sign is to be taken when the inner surface of the shell is positive, and the minus sign when the inner surface is negative. Accordingly the magnetic force is zero both inside and outside, because the potential is in each case constant.

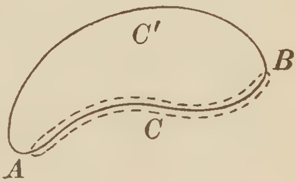
The general case of a shell which is not closed is rather complicated, on account of the possibility of edges of the shell, portions of its contour, overlapping from some points of view, and not from others. Draw a sphere of unit radius with centre at P. And let a radius from P move continuously along the contour, tracing out a curve on the sphere, which may have points of intersection; and

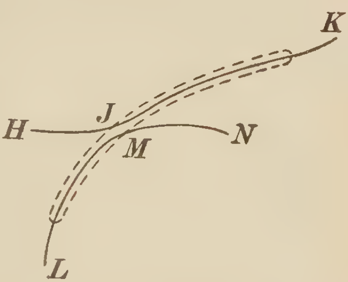
mark on this curve with arrow heads the way the radius moves. If the curve is a single loop, it divides the area of the sphere into two parts the sum of which is 4π. Thus we have two solid angles, Ω and $4\pi - \Omega$; and one of them corresponds to radii which cut the shell an odd number of times, and multiplied by ϕ or $-\phi$ can be distinguished as the potential of the shell at P. But if intersections occur they indicate overlap. The arrow heads show whether there is, or is not, an intersection where two loops join. In the diagram, taking the point P to be behind the paper, two loops are traversed so as to make a right-handed screw with the direction of the radius, and one so as to make a left-handed screw with this direction. And the effective solid angle, for the calculation of the potential at P, is the difference between the sum of right-handed areas and the sum of left-handed areas. This rule can be verified by examination of the way in which solid angles cancel one another in particular cases. And finally the sign of the resulting potential depends on which side of the shell is positive. In studying the curve traced on the sphere, loops which overlap must be taken into account separately.

This shows that, with a given strength, and a given contour, the form of the shell may be gradually altered, without altering the potential at P, provided that P remains on the same side of it. But if, as the form of the

shell is gradually altered, it reaches P and passes to the other side of it, the value of the potential at P is abruptly altered by the addition or subtraction of $4\pi\phi$. This is most easily seen by supposing the shell to be fixed, and the point P to be taken first on the negative face of it, and then on the positive face of it. The difference of potential at P, due to this infinitesimal change, must depend only on the effect of a small portion of the shell at the place where P passes through it, the potential due to which changes from $-2\pi\phi$ to $+2\pi\phi$.

The character of the equipotential surfaces of magnetic force due to a shell can be seen by taking the simple case of a shell whose contour is a single loop in one plane. The diagram represents a plane section of a shell and of one of the surfaces. The shell, AB, is shown by dotted lines; A and B being points on the contour. And $ACBC'A$ is the section of one of the surfaces. The surface is of course a closed surface. The part of it outside the shell is the locus of points at which the contour of the shell subtends a certain constant solid angle. And the remainder is sufficiently defined by the fact that it is inside the shell. All the surfaces can be drawn in this way, all of them crowded into the substance of the shell in such a way that none of them intersect. A surface may be such that a portion, HJ, outside the shell, meets it at some point J. In that case it passes inside the shell and is continued along the line JK; and at the same time a surface, LM, inside the shell leaves it and follows the line MN, such that $HJMN$ is a surface specified by the solid





angle. This is necessary because the number of surfaces inside the substance of the shell is the same at all points of it, namely $4\pi\phi$.

Now we know that B is the same as H outside the shell. Also we know that surfaces specified by the solid angle depend only on the contour, and are independent of the form of the shell in other respects. Therefore B is represented by the surfaces specified by the solid angle, the portions of surface inside the shell being abolished. And the lines of B, which has a solenoidal distribution, are lines drawn so as to cut these surfaces at right angles. They are loops which thread the contour of the shell.

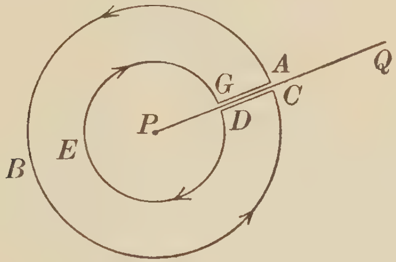
234. *Electric Currents.* The well known law of the magnetic force due to an electric current, C, in a given circuit, is that it is the same as the magnetic induction due to a magnetic shell of strength kC, whose contour is the given circuit; where k is a constant. The electromagnetic unit of current is chosen so as to make k equal to 1.

In the case of an electric current, C, in a straight linear conductor of infinite length, any plane drawn from this straight line to infinity may be taken to represent the corresponding shell. Therefore the lines of induction are loops encircling the conductor. Also the symmetry of the system shows that these loops are circles, in planes at right angles to the conductor, with their centres on that line; and that the magnetic force in each loop must be constant.

In a plane at right angles to the conductor draw two of these loops, and denote their radii by a and b, and the magnetic force in them due to the current by F and F'. Now the circulation of magnetic force in any circuit in the region outside the shell is zero. Let us take the circuit $ABCDEGA$ consisting of the two circles and straight lines CD, GA drawn on the two faces of the shell, so that the shell is not cut. Here P is the conductor, and PQ is the

shell. Equating the circulation in this circuit to zero, we get

$$2\pi aF - 2\pi bF' = 0.$$



This shows that the magnetic force due to the current is inversely proportional to the distance from the conductor. But $2\pi aF$ is equal to the work done on a unit pole in going through a shell of strength C, which is $4\pi C$. Therefore $F = \frac{2C}{a}$. Thus the magnetic force due to the current, at a distance r from it, is $\frac{2C}{r}$.

The direction of the magnetic force along a line ABC is from the positive side of the shell towards the negative side; and the direction of the current is found to be that which makes a right-handed screw with this way round the circle.

This shows that the direction of a current in a linear conductor, in the form of a loop, makes a right-handed screw with the way the magnetic force points in the lines of force threading it. And it follows from this that the direction of magnetization of an electromagnet makes a right-handed screw with the direction of the exciting current.

CHAPTER XVIII

APPENDIX

235. *Notes.*

1. It is important to notice that a considerable part of Chapter IX deals with projective properties of a figure, with regard to which it is immaterial whether the coordinate axes are rectangular or oblique; so that, if the term direction cosines, and references to normals and right angles, are omitted, the coordinate axes may be oblique. This liberty ceases when principal planes are introduced.

An obvious case for the use of oblique axes is that in which two or more quadric surfaces have a pair of intersecting generating lines in common. Taking these lines as axes of x and y, the general equation of these surfaces must be satisfied by $z = 0$, $x = 0$, and by $z = 0$, $y = 0$, and is therefore

$$cz^2 + 2fyz + 2gzx + 2hxy + 2wz = 0.$$

The common normal may be taken for axis of z. The centre is given, as in § 120, by the equations

$$hy + gz = 0,$$
$$hx + fz = 0,$$
$$gx + fy + cz + w = 0.$$

And if the quadrics are subject to some further conditions, the locus of the centre may be found.

If the direction of the diameter through the origin is given, this may be taken for axis of z. The equations for the centre then show that g and f must be zero, so that the general equation of the surfaces in question is

$$cz^2 + 2hxy + 2wz = 0;$$

and the coordinates of the centre are $(0, 0, -w/c)$.

2. This note is a supplement to § 119, dealing with the conditions for the surface $F(xyz) = 0$ being a cone or a cylinder. In either case we have

$$-(\mathbf{A}u^2 + \mathbf{B}v^2 + \mathbf{C}w^2 + 2\mathbf{F}uv + 2\mathbf{G}wu + 2\mathbf{H}uv) + \Delta d = 0,$$

where $\mathbf{A} = bc - f^2$, $\mathbf{B} = ca - g^2$, $\mathbf{C} = ab - h^2$, $\mathbf{F} = gh - af$, $\mathbf{G} = hf - bg$, $\mathbf{H} - fg - ch$. And the cases in which the surface is a cylinder are those in which the point P, (§ 118), passes to infinity, and that in which the surface is a cone which is also a cylinder, (§ 96). These cases are distinguished by

Δ being zero. Accordingly the surface is either a single point, or a cone with a single vertex, if the condition written above is satisfied and Δ is not zero; and the condition for the surface being a cylinder is given by the pair of equations

$$\Delta = 0, \quad \mathbf{A}u^2 + \mathbf{B}v^2 + \mathbf{C}w^2 + 2\mathbf{F}uv + 2\mathbf{G}wu + 2\mathbf{H}uv = 0.$$

Now

$$\mathbf{BC} - \mathbf{F}^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$$
$$= a(abc + 2fgh - af^2 - bg^2 - ch^2) = a\Delta;$$

and similarly

$$\mathbf{CA} - \mathbf{G}^2 = b\Delta, \quad \mathbf{AB} - \mathbf{H}^2 = c\Delta, \quad \mathbf{GH} - \mathbf{AF} = f\Delta,$$
$$\mathbf{HF} - \mathbf{BG} = g\Delta, \quad \mathbf{FG} - \mathbf{CH} = h\Delta.$$

Therefore, for a cylinder, Δ being zero,

$$\mathbf{AF} = \mathbf{GH}, \quad \mathbf{BG} = \mathbf{HF} \quad \text{and } \mathbf{CH} = \mathbf{FG}.$$

Therefore $\quad \mathbf{FGH}(\mathbf{A}u^2 + \mathbf{B}v^2 + \mathbf{C}w^2 + 2\mathbf{F}uv + 2\mathbf{G}wu + 2\mathbf{H}uv)$

is a perfect square, namely $\quad (\mathbf{GH}u + \mathbf{HF}v + \mathbf{FG}w)^2$.

This shows that the condition for the surface being a cylinder is given by the pair of equations

$$\Delta = 0, \quad \mathbf{GH}u + \mathbf{HF}v + \mathbf{FG}w = 0,$$

provided that **FGH** is not zero. But **FGH** being zero cannot deprive $\Phi(u, v, w)$ of the property of being a perfect square, accordingly the equation to be satisfied by u, v and w is in all cases a linear one.

3. The surface represented by the equation

$$\frac{a^2x^2}{a^2 - r^2} + \frac{b^2y^2}{b^2 - r^2} + \frac{c^2z^2}{c^2 - r^2} = 0,$$

where r^2 is written for $x^2 + y^2 + z^2$, is called Fresnel's Wave surface. By referring to § 78 it will be seen that, on each radius drawn from the origin, there are two and only two points on this surface; namely the points whose distances from the origin are r_1 and r_2, where r_1, r_2 are the lengths of the semi-axes of the section of the ellipse,

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

by the plane through the origin at right angles to this radius; subject to the exception that the two points coincide where the section is a circle.

Accordingly the surface consists of two sheets embracing the origin. And these sheets meet at four points in the plane of the greatest and least axes of the ellipsoid. These are the famous conical points, the existence of which, (Sir W. R. Hamilton having called attention to them), led to the discovery of conical refraction of light in a crystal.

The equation written at length is

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - \{a^2x^2(b^2 + c^2) + b^2y^2(c^2 + a^2) + c^2z^2(a^2 + b^2)\} + a^2b^2c^2 = 0.$$

From this equation it will be seen that the section by each of the coordinate planes consists of an ellipse and circle. For example the equation of the section by the plane $z = 0$ is

$$(x^2 + y^2 - c^2)(a^2x^2 + b^2y^2 - a^2b^2) = 0.$$

In the plane at right angles to the mean axis of the ellipsoid, the ellipse and the circle intersect.

4. The definition of a geodesic on a surface, (§ 174), namely that it is a curve whose principal normal coincides with the normal of the surface, enables us to calculate the curvature, $1/\rho$, and the torsion, $1/\sigma$, of a geodesic at any given point, P, at which it cuts a line of curvature at a given angle.

The equation of the surface, in the vicinity of P, is

$$2z = x^2/\rho_1 + y^2/\rho_2 + \text{higher powers},$$

the axes being the tangents of the lines of curvature at P and the normal; and $1/\rho_1$, $1/\rho_2$ being the curvatures of the principal normal sections. Let ψ be the angle at which the geodesic cuts the axis of x. The direction cosines of the normal at P are $(0, 0, 1)$. And the equation of the surface gives by differentiation

$$p = x/\rho_1 + \ldots\ldots, \qquad q = y/\rho_2 + \ldots\ldots,$$

therefore, (§ 168), the direction cosines of the normal at a point on the curve at distance ds from P are $\left(-\dfrac{ds\cos\psi}{\rho_1},\ -\dfrac{ds\sin\psi}{\rho_2},\ 1\right)$. Accordingly, with the notation of § 150, the direction cosines (l, m, n), (l', m', n') and (λ, μ, ν) for the geodesic, at the point P, are

$$(\cos\psi, \sin\psi, 0), \quad (0, 0, 1) \quad \text{and} \quad (\sin\psi, -\cos\psi, 0);$$

and

$$\frac{dl'}{ds} = -\frac{\cos\psi}{\rho_1}, \quad \frac{dm'}{ds} = -\frac{\sin\psi}{\rho_2}.$$

Therefore Frenet's formulae, (§ 153), give

$$-\frac{\cos\psi}{\rho_1} = -\frac{l}{\rho} - \frac{\lambda}{\sigma} = -\frac{\cos\psi}{\rho} - \frac{\sin\psi}{\sigma},$$

$$-\frac{\sin\psi}{\rho_2} = -\frac{m}{\rho} - \frac{\mu}{\rho} = -\frac{\sin\psi}{\rho} + \frac{\cos\psi}{\sigma}.$$

And solving these equations,

$$\frac{1}{\rho} = \frac{\cos^2\psi}{\rho_1} + \frac{\sin^2\psi}{\rho_2}, \quad \frac{1}{\sigma} = \sin\psi\cos\psi\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

It should be noticed that, for a geodesic at right angles to this one, the torsion has the same value with opposite sign.

For a curve on the surface, tangential to the geodesic, but with its osculating plane inclined at an angle ϕ to the normal, the curvature is given by Meunier's theorem, and the torsion by the equation

$$\frac{1}{\sigma} + \frac{d\phi}{ds} = \sin\psi\cos\psi\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

5. To make the construction referred to in § 194, we take the principal axes at the centroid, G, for coordinate axes, and draw the ellipsoid of gyration, $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, at G. And it must be assumed that we can draw the quadrics confocal to this ellipsoid, each of them specified by its parameter, the equation of the confocal with parameter λ being

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

It is obvious that one and only one of these confocals can be drawn to touch a given plane, $lx + my + nz = p$, because the condition for tangency is a simple equation for λ, namely,

$$(a^2+\lambda)\, l^2 + (b^2+\lambda)\, m^2 + (c^2+\lambda)\, n^2 = p^2.$$

Let I be the moment of inertia at any point, P, (x', y', z'), about a line PQ with direction cosines (l, m, n). And let p be the perpendicular from G on a plane drawn through P at right angles to PQ, and q the perpendicular distance from G of the line PQ. Then, if λ is the parameter of the confocal which touches the plane drawn through P at right angles to PQ,

$$I = M(a^2l^2 + b^2m^2 + c^2n^2) + Mq^2 = M(p^2 - \lambda + q^2) = M(GP^2 - \lambda).$$

To find the principal axes at the given point, P, (x', y', z'), we have to find the directions, (l, m, n), for which I is greatest and least, that is to say for which λ is least and greatest. To do this we have the equation

$$(a^2+\lambda)\, l^2 + (b^2+\lambda)\, m^2 + (c^2+\lambda)\, n^2 = (lx' + my' + nz')^2,$$

or

$$\lambda = (x'^2 - a^2)\, l^2 + (y'^2 - b^2)\, m^2 + (z'^2 - c^2)\, n^2 + 2y'z'mn + 2z'x'nl + 2x'y'lm.$$

Accordingly the greatest and least values of λ are given by the discriminating cubic, (§ 99), of the righthand side of this equation, regarded as a quadratic function of l, m, and n; and this proves on examination to be the equation

$$\frac{x'^2}{a^2+k} + \frac{y'^2}{b^2+k} + \frac{z'^2}{c^2+k} = 1,$$

the roots of which are the parameters, say λ_1, λ_2, λ_3, of the three confocals which intersect at P. Therefore the principal moments of inertia at P are

$$M(GP^2 - \lambda_1), \quad M(GP^2 - \lambda_2), \quad M(GP^2 - \lambda_3),$$

and the normals at P of these confocals are the principal axes at this point.

236. *Miscellaneous Examples.*

1. Find the equation of the plane which contains the origin and the straight line

$$x + 2y + 3z + 4 = 2x + 3y + 4z + 1 = 3x + 4y + z + 2.$$

2. Find the equation of a plane equally inclined to the three axes, and containing the point of intersection of the planes

$$x + 2y - 3z = 1, \quad 2x - 3y + 5z = 3, \quad 7x - y - z = 2.$$

3. Prove that the condition that the equations

$$a + mz - ny = 0, \quad b + nx - lz = 0, \quad c + ly - mx = 0,$$

represent a straight line is $al + bm + cn = 0$.

4. Show that the equation of the plane which is the reflection of the plane $ax + by + cz + d = 0$, with respect to the plane

$$Ax + By + Cz + D = 0,$$

is

$$(A^2 + B^2 + C^2)(ax + by + cz + d) - 2(Aa + Bb + Cc)(Ax + By + Cz + D) = 0.$$

(C.)

5. Two straight lines being given, a plane is drawn through each of them. Prove that, if the planes are at right angles, their intersection, in general, traces out a hyperboloid of one sheet. What are the exceptions? (C.)

6. Find the equation of the plane which contains the straight line $(x - a)/l = (y - b)/m = (z - c)/n$, and is perpendicular to the plane $Ax + By + Cz + D = 0$.

7. Find the equations of the three planes whose lines of intersection are parallel, and equally inclined to the coordinate axes, and meet the plane of xy in the points $(1, 2, 0)$, $(3, 1, 0)$, $(2, 3, 0)$.

8. Show that the orthogonal projection of the line, whose six coordinates are $(l, m, n, \lambda, \mu, \nu)$, on the plane $Ax + By + Cz = 0$, is the intersection of this plane with

$$A(\lambda - ny + mz) + B(\mu - lz + nx) + C(\nu - mx + ly) = 0. \qquad \text{(S.)}$$

9. Find the radii of the spheres which touch all the coordinate planes and the plane $x + y + z = a$.

10. Prove that if a sphere cuts orthogonally two spheres, whose equations are $S = 0$ and $S' = 0$, it will cut orthogonally the sphere $S + \lambda S' = 0$. (S 1.)

11. Through two given straight lines are drawn two planes, so as to be equally inclined to a given plane parallel to both the lines. Prove that the line of intersection of the two variable planes must lie on one of two fixed paraboloids. (C.)

12. Prove that the radius of a circular section of the ellipsoid
$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \quad (a > b > c),$$
at a distance p from the centre, is $b\left(1 - \dfrac{p^2 b^2}{a^2 c^2}\right)^{\frac{1}{2}}$. (S 1.)

13. Show that there are, in general, two straight lines which meet four given non-intersecting straight lines. If the four straight lines are
$$y = z = 0; \quad y = x,\ z = c; \quad y = -x,\ z = -c; \quad y = mx,\ z = nx + mc;$$
show that the two transversals coincide, and find their equations. (C.)

14. For a hyperboloid of one sheet, prove that the product of the sines of the angles that a generator makes with the planes of the circular sections is constant. (C.)

15. Find the equation of the surface generated by the revolution of the line
$$\frac{x-a}{A} = \frac{y-b}{B} = \frac{z-c}{C} \text{ about the line } \frac{x-a'}{A'} = \frac{y-b'}{B'} = \frac{z-c'}{C'}. \quad \text{(C.)}$$

16. Show that any two cones, drawn so as to envelop a given central quadric, intersect one another in plane curves. (S.)

17. Prove that the normals of the ellipsoid, $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, at the points (x', y', z'), (x'', y'', z'') will intersect if
$$(b^2 - c^2)\frac{x'}{x'' - x'} + (c^2 - a^2)\frac{y'}{y'' - y'} + (a^2 - b^2)\frac{z'}{z'' - z'}, = 0;$$
and that, if (X, Y, Z) is the point of intersection,
$$a^2 X\left(\frac{1}{x'} - \frac{1}{x''}\right) = b^2 Y\left(\frac{1}{y'} - \frac{1}{y''}\right) = c^2 Z\left(\frac{1}{z'} - \frac{1}{z''}\right). \quad \text{(S 1.)}$$

18. Normals are drawn at all points of the central circular sections of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $(a > b > c)$. Prove that they intersect the plane $x = 0$ upon the ellipse
$$\frac{b^2 y^2}{a^2 - b^2} + \frac{c^2 z^2}{a^2 - c^2} = a^2 - b^2. \quad \text{(S 1.)}$$

19. Show that if three equal conjugate diameters of the spheroid,
$$x^2/a^2 + (y^2 + z^2)/b^2 = 1,$$
are drawn, the cosine of the angle between any two of them is
$$(a^2 - b^2)/(a^2 + 2b^2). \quad \text{(S 1.)}$$

20. Find the locus of the point of intersection of tangent planes at the extremities of a set of conjugate diameters of a given ellipsoid.

21. Find the condition that the section of the quadric

$$ax^2 + by^2 + cz^2 = 1$$

by the plane

$$lx + my + nz = p$$

is a parabola; and find the direction cosines of its axis. (C.)

22. Prove that the condition that the section of the quadric, $ax^2 + by^2 + cz^2 = 1$, by the plane $lx + my + nz = 0$, is a rectangular hyperbola is

$$(a + b + c)(l^2 + m^2 + n^2) = al^2 + bm^2 + cn^2. \quad \text{(C.)}$$

23. Find the equation of the polar reciprocal of the quadric, $Ax^2 + By^2 + Cz^2 = 1$, with respect to the paraboloid $ax^2 + by^2 = 2z$. (C.)

24. Show that if the equation of a hyperboloid of one sheet is written

$$ax^2 + by^2 + cz^2 + abc = 0,$$

and $al^2 + bm^2 + cn^2 = 0$, the line (l, m, n, al, bm, cn) is a generator. (C.)

25. Find the generating lines of the surface $3zx - y - z + 1 = 0$ at the point $(0, 0, 1)$.

26. A hyperbolic paraboloid is referred to any point on the surface as origin, and the generators there as axes of x and y; show that the co-ordinates of any other point on the surface may be expressed by

$$x = p(1 + fq), \quad y = q(1 + gp), \quad z = hpq;$$

where p and q are the intercepts made on the axes by the generators through the point, and f, g, h are constants. (C.)

27. Calculate the lengths of the principal axes of the section of the quadric,

$$3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 20,$$

by the plane

$$x + y + z = 0.$$

28. Show that the envelope of the plane sections of the system of confocal quadrics,

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1,$$

which have their centre at a fixed point, (α, β, γ), is the quadric cone whose equation referred to axes through its vertex is

$$\sqrt{\alpha x (b^2 - c^2)} + \sqrt{\beta y (c^2 - a^2)} + \sqrt{\gamma z (a^2 - b^2)} = 0. \quad \text{(C.)}$$

29. A point moves so that its distances from two given straight lines are in a constant ratio. Prove that its locus is, in general, a hyperboloid of one sheet, such that the sum of the reciprocals of the squares of its principal axes is equal to the reciprocal of the square of the principal axis of its conjugate. (S.)

30. Prove that the surface

$$x = \lambda \cos\theta - a\sin\theta, \quad y = \lambda\sin\theta + a\cos\theta, \quad z = b\theta,$$

where λ and θ are parameters, is a ruled surface. (C.)

31. Find the equations of the two systems of curves which are the projections on the plane of xy of the lines of curvature of the surface $az = xy$. (C.)

32. Prove that if ρ, ρ', n are the principal radii of curvature and the normal chord at any point of the surface $az = xy$, then

$$\frac{1}{\rho} + \frac{1}{\rho'} + \frac{2}{n} = 0. \qquad \text{(S.)}$$

33. Show that at a point, (x, y, z), on the surface

$$x^4 + y^4 + z^4 = a^4,$$

the product of the principal radii of curvature is

$$\left\{\frac{x^6 + y^6 + z^6}{3a^2xyz}\right\}^2. \qquad \text{(C.)}$$

34. Find the polar developable of the curve $x^2 = az$, $x = y$.

35. Prove that at every point on the surface

$$z = (y - b)\tan x/a$$

the sum of the principal curvatures is zero, and calculate their product at any given point.

36. Find the locus of the feet of the perpendiculars from a given point, (x', y', z'), on the generators of the conoid $z = f(\theta)$, where $\tan\theta = y/x$; and prove that the condition that it is a plane curve is

$$f'''(\theta) + 4f'(\theta) = 0,$$

(the dashes denoting differentiation). Integrate this equation, and show that the surface obtained is a cylindroid, (§ 175). (C.)

37. Show that the tangent of the locus of the centre of curvature of a curve is inclined to the principal normal at an angle ϕ, such that $\cot\phi = \frac{\sigma}{\rho}\frac{d\rho}{ds}$; and that the angle between adjoining positions of this line is

$$\left\{\left(\frac{\cos\phi}{\rho}\right)^2 + \left(\frac{1}{\sigma} + \frac{d\phi}{ds}\right)^2\right\}^{\frac{1}{2}} ds.$$

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